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A New Construction of Polarizations of Solvable Lie Algebras

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Abstract

Let G be a real solvable Lie algebra and G^* the vector space dual. For $\phi \in G^*$, a construction of a polarization at ϕ is described. The construction is based on the use of maximal elements of G with respect to a partial order that depends on ϕ. It is proved that the construction breaks down only when G fails to satisfy the criterion of Brezin and Dixmier.

Introduction

Let G be a real solvable Lie algebra and G^*, the vector space dual. For $\phi \in G^*$, one is interested in producing a *polarization* of G at ϕ. That is, a subalgebra H with $\phi[H, H] = 0$ and $dimH = \frac{1}{2}(dimG + dimR_\phi)$, where $R_\phi = \{x \in G : \phi[x, y] = 0 \text{ for all } y \in G\}$ is what we shall call the *radical* of ϕ. In this paper, a construction of polarizations is described. The construction is based on the use of maximal elements (or ϕ-extreme points) of G with respect to a partial order that depends on ϕ.

Vergne [5] has given an elegant construction of polarizations. By comparison, the "extreme point construction" is more difficult to use but appears to yield more polarizations, as the example at the end of the paper will show.

The following provides a starting point in describing the structure of a real solvable Lie algebra:

Definition 1 *A series of ideals*

$$G = H_0 \supset H_1 \supset H_2 \supset \ldots \supset H_n = 0$$

of a Lie algebra G *is said to be a chief series of* G *if* H_i/H_{i+1} *is a minimal ideal of* G/H_{i+1} *for every* $0 \leq i \leq n-1$.

Note. Every (finite dimensional) Lie algebra admits a chief series of ideals.

Let G be a real solvable Lie algebra and let $G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_n = 0$ be a chief series of G. It must be that the adjoint representation of G on H_i/H_{i+1} is irreducible for each i (or else H_i/H_{i+1} would not be a minimal ideal of G/H_{i+1}). Also, since the minimal ideals of a real solvable Lie algebra always have dimension 1 or 2 ([3], p.8), the codimension of H_{i+1} in H_i is either 1 or 2 for each i. It follows from irreducibility that, when the codimension is 2, there exists a pair of elements $x, y \in H_i$, linearly independent mod H_{i+1} , such that for all $t \in G$

$$[t, x] = \alpha(t)x - \beta(t)y \qquad (\text{mod } H_{i+1})$$

and

$$[t, y] = \beta(t)x + \alpha(t)y \qquad (\text{mod } H_{i+1}),$$

where α and β are linear forms on G which are zero on $[G, G]$ and β is not identically zero. If, instead, the codimension is 1 then for any nonzero $x \in H_i$ with $x \notin H_{i+1}$ and for all $t \in G$ we have

$$[t, x] = \gamma(t)x \qquad (\text{mod } H_{i+1}),$$

where γ is a linear form on G which is zero on $[G, G]$.

ϕ-extreme points

Let G be a real solvable Lie algebra, fix $\phi \in G^*$, and let R be the radical of ϕ. For any $x \in G$, let

$$G(x) = \mathbb{R}x \cup [G, x] \cup [G, [G, x]] \cup \cdots.$$

Note that this set is closed under multiplication by scalars, is closed under the operations of adG, and is contained in every ideal containing x.

Definition 2 *For any subset S of G we will say that a nonzero element $x \in S$ is ϕ-extreme in S provided that $\phi[u, v] = 0$ for all $u, v \in G(x) \cap S$.*

The idea of ϕ-extreme points comes from a partial order one can define on G. For fixed ϕ and for any x and y in G, let us say that $x < y$ if and only if $y \in G(x)$ and $\phi[y, z] \neq 0$ for some $z \in G(x)$. This relation is easily seen to be transitive (and therefore it is a partial order) but not generally anti-symmetric or reflexive. An element x in the domain or range of this partial order is maximal with respect to the partial order if and only if x is ϕ-extreme in G.

Proposition 1 *Let G be a real solvable Lie algebra with chief series $G = H_0 \supset H_1 \supset \cdots \supset H_n = 0$. Let ϕ be a nontrivial element of G^* and let $S = G \setminus R$. Choose i so that $S \cap H_i \neq \emptyset$ but $S \cap H_{i+1} = \emptyset$. If the codimension of H_{i+1} in H_i is 1 then the elements of $S \cap H_i$ are ϕ-extreme points of S.*

Proof. Let $x \in S \cap H_i$. If x is not a ϕ-extreme point of S then there exist $y, z \in G(x) \cap S \subset H_i$ with $\phi[y, z] \neq 0$. Since $x \notin H_{i+1}$ and the codimension of H_{i+1} in H_i is 1, y and z can be expressed as $y = \alpha x + u$ and $z = \beta x + v$ for some $u, v \in H_{i+1}$ and $\alpha, \beta \in \mathbb{R}$. But then since $u, v \notin S$, they are in the radical of ϕ and $\phi[y, z] = \phi[\alpha x + u, \beta x + v] = 0$, which is a contradiction.

Construction of polarizations

We want to produce a subalgebra H such that $\phi[H, H] = 0$ and $dimH = \frac{1}{2}(dimG + dimR)$. First, it must be recalled that such a subalgebra does not always exist.

Example Let D be the *diamond* Lie algebra whose basis over $\mathbb{R}$ is $\{t, x, y, z\}$ with the nonzero brackets among basis elements being

$$[t, x] = -y, [t, y] = x, [x, y] = z.$$

Let ϕ be defined by $\phi(z) = 1, \phi(t) = \phi(x) = \phi(y) = 0$. The dimension of R is 2 but there is no subalgebra of D subordinate to ϕ having dimension $\frac{1}{2}(4 + 2)$. In fact, one has the criterion of Brezin ([1], theorem 32) and Dixmier:

Theorem 1 *Let G be a real solvable Lie algebra and $\phi \in G^*$. If there does not exist a polarization of G at ϕ then G has the diamond property. That is, there exists a subalgebra K of G and an ideal J of K with the property that K/J is isomorphic to D.*

We will find in Theorem 4 that the following construction of polarizations fails only when G has the diamond property. Begin the construction by letting $S = G \setminus R$. If S is empty then $G = H = R$ is a polarization at ϕ and we are done. If not, let y_0 be a ϕ-extreme point of S (if no such point exists then the construction fails; see Theorem 4) and define

$$G_1 = \{x \in G : \phi[x, z] = 0 \text{ for all } z \in G(y_0) \cap S\}.$$

Note that $y_0 \in G(y_0) \cap S \subset G_1$ since y_0 is ϕ-extreme in S. Also, because $y_0 \in S$, there exists $x_0 \in G$ with $\phi[x_0, y_0] \neq 0$ so $x_0 \notin G_1$ and we have $G_1 \neq G$.

The set G_1 is a subalgebra of G; it is seen to be closed under the bracket operation by the following argument: let $u, v \in G_1$ and $z \in G(y_0) \cap S$. By the Jacobi identity,

$$\phi[[u, v], z] = -\phi[[z, u], v] - \phi[[v, z], u].$$

Since z is in $G(y_0)$, so are $[z, u]$ and $[v, z]$; and each of these is either in S or not. Either way the definitions indicate that both terms on the right hand side of the equation above are zero. Thus, $[u, v] \in G_1$.

Let $R_1 = \{x \in G_1 : \phi[x, y] = 0 \text{ for all } y \in G_1\}$. This is the radical of the restriction ϕ_1 of ϕ to G_1. It will be shown in Theorem 2 that

$$\frac{1}{2}(dimG + dimR) = \frac{1}{2}(dimG_1 + dimR_1),$$

so if we find a polarization of G_1 at ϕ_1, it will also be a polarization of G at ϕ. If $R_1 = G_1$ then set $H = G_1$ and we are done. Otherwise the procedure is repeated using G_1 and ϕ_1 in place of G and ϕ. In this way, the subalgebras $G_1, G_2, ..., G_m$ are recursively defined. For each i with $1 \leq i \leq m - 1$,

$$G_{i+1} = \{x \in G_i : \phi_i[x, z] = 0 \text{ for all } z \in G_i(y_i) \cap S_i\},$$

where ϕ_i is the restriction of ϕ to G_i, $S_i = G_i \setminus R_i$, R_i is the radical of ϕ_i, and y_i is a ϕ_i-extreme point of S_i. The process ends when $R_m = G_m$. This subalgebra is then a polarization of G at ϕ since $\phi[G_m, G_m] = 0$ and $\frac{1}{2}(dimG + dimR) = \frac{1}{2}(dimG_m + dimR_m) = dimG_m$ by the following:

Theorem 2 *There exists a sequence of pairs x_i, z_i of elements of G for $i = 1, ..., k$ such that $\phi[x_i, z_j] \neq 0$ if and only if $i \neq j$,*

$$G = G_1 \oplus span\{x_1, ..., x_k\},$$

and

$$R_1 = R \oplus span\{z_1, ..., z_k\}.$$

Thus,

$$\frac{1}{2}(dimG + dimR) = \frac{1}{2}(dimG_1 + dimR_1).$$

Proof. We assume that y_0 is a ϕ-extreme point of $S = G \setminus R$ and then $G_1 = \{x \in G : \phi[x, z] = 0 \text{ for all } z \in G(y_0) \cap S\} = \cap ker(\phi \circ adz)$, the intersection being taken over all $z \in G(y_0) \cap S$. Since the codimension of $ker(\phi \circ adz)$ in G is 1 for every $z \in S$, we can choose $k = dimG - dimG_1$ elements $z_1, \ldots, z_k \in G(y_0) \cap S$ so that

$$G_1 = ker(\phi \circ adz_1) \cap \ldots \cap ker(\phi \circ adz_k)$$

but G_1 is not the intersection of fewer than k of these sets. For each $i = 1, ..., k$ choose $x_i \in (\cap ker(\phi \circ adz_j)) \setminus G_1$, the intersection being taken over all $j \neq i$.

In order to compute R_1, note that if $x \in R$ then $\phi[x, y] = 0$ for all $y \in G$, so $x \in G_1$ and $x \in R_1$. Thus, $R \subset R_1$. Using the fact that $G(y_0) \cap S \subset G_1$ (since y_0 is a ϕ-extreme point of S) we find that the elements $z_1, ..., z_k$ are in G_1. Moreover, they are in R_1 because $G_1 \subset ker(\phi \circ adz_i)$ for $i = 1, ..., k$. But they are not in R since $\phi[z_i, x_i] \neq 0$. By repeatedly using the fact that $\phi[x_i, z_j]$ is nonzero if and only if $i \neq j$, one shows that G and R_1 are vector space direct sums: $G = G_1 \oplus span\{x_1, ..., x_k\}$ and $R_1 = R \oplus span\{z_1, ..., z_k\}$.

For our fixed $\phi \in G^*$ and any subalgebra H of G, consider the following property which is of interest in the theory of representations of Lie groups (see Vergne [5]):

$(*)$ *If* $\alpha \in G^*$ *with* $\alpha(H) = 0$*, then* $H^{\alpha+\phi} = H$*, where* $H^{\alpha+\phi}$ *denotes the subspace of* G *orthogonal to* H *relative to the alternating bilinear form* $(\alpha + \phi)[\cdot, \cdot]$.

Theorem 3 *The subalgebra* $H = G_m = R_m$ *arising from the extreme point construction satisfies property (*).*

Proof. Let $\alpha \in G^*$ with $\alpha(H) = 0$. We want to prove that $H^{\alpha+\phi} = \{x \in G : (\alpha + \phi)[x, y] = 0 \text{ for all } y \in H\} = H$. Assume $x \in H^{\alpha+\phi}$ and suppose we have $x \in G_i$ but $x \notin G_{i+1}$ for some $0 \leq i \leq m - 1$ (letting $G_0 = G$ in case $i = 0$). Then by the definition of G_{i+1}, there exists $z \in G_i(y_i) \cap S_i$ such that $\phi[x, z] \neq 0$. Since $z \in G_i(y_i)$ and $x \in G_i$, we find that $[x, z] \in G_i(y_i) \subset R_{i+1} \subset R_m = H$ (see the proof of Theorem 2) and, hence, $\alpha[x, z] = 0$. On the other hand, it also follows from the definition of G_{i+1} that $\phi[G_{i+1}, z] = 0$ so that $z \in R_{i+1} \subset R_m = H$ and, by our assumption, $(\alpha + \phi)[x, z] = 0$. Therefore $\phi[x, z] = 0$, which is a contradiction, and it follows that $x \in G_m = H$.

The converse is clear.

Remark Theorem 3 could have been used instead of Theorem 2 to show that the extreme point construction yields a subalgebra $H = G_m$

having the maximal dimension. This is because any subalgebra H satisfying property (*) is both subordinate to ϕ, since $H \subset H^{\phi}$, and has the maximal dimension, since $H^{\phi} \subset H$ ([2], Section 1.12).

Theorem 4 *If the extreme point construction fails for some $\phi \in G^*$, then G has the diamond property.*

Proof. If the extreme point construction fails at some stage it means there is a subalgebra G_k of G so that the complement of the radical of $\phi|G_k$ fails to have a $\phi|G_k$ -extreme point. Since it is enough to show that G_k has the diamond property, there is no loss in generality to assume that G_k is G. That is, we assume that the construction fails in the first stage.

Let $G = H_0 \supset H_1 \supset ... \supset H_n = \{0\}$ be a chief series for G. For the fixed element $\phi \in G^*$, we have $S = \{x \in G : \phi[x, y] \neq 0$ for some $y \in G\}$. Let i be chosen so that $S \cap H_i \neq \emptyset$ but $S \cap H_{i+1} = \emptyset$. This says that ϕ is identically zero on $[G, H_{i+1}]$ but not on $[G, H_i]$. By Proposition 1, the codimension of H_{i+1} in H_i must be 2 since otherwise the elements of $S \cap H_i$ would be ϕ-extreme points. Thus, there exist x and $y \in H_i$, linearly independent mod H_{i+1} , so that for all $t \in G$

$$\begin{aligned} [t, x] &= \alpha(t)x - \beta(t)y \qquad (\text{mod } H_{i+1}) \\ [t, y] &= \beta(t)x + \alpha(t)y \qquad (\text{mod } H_{i+1}) \end{aligned} \tag{1}$$

where α and β are linear forms on G which are zero on $[G, G]$ and β is not identically zero. Fix t so that $\beta(t) \neq 0$ and let $\alpha = \alpha(t)$ and $\beta = \beta(t)$.

Let $J = \{w \in H_{i+1} : \phi(w) = 0\}$. This is an ideal of G because ϕ vanishes on $[G, H_{i+1}]$. Now define K to be the subspace of G spanned by $\{t, x, y, [x, y]\} \cup J$. It turns out that K is a subalgebra and K/J is isomorphic to D, the diamond Lie algebra. In order to see this, we can construct elements T, X, Y, Z of K, linearly independent mod J, so that

$$[T, X] = -Y, \quad [T, Y] = X, \quad \text{and} \quad [\mathrm{X}, \mathrm{Y}] = \mathrm{Z} \qquad (\text{mod J})$$

and

$$[Z, X] = [Z, Y] = [Z, T] = 0 \qquad (\text{mod } J).$$

First note that, since $\{x, y\}$ is a basis for $H_i \pmod{H_{i+1}}$ and ϕ vanishes on $[G, H_{i+1}]$, it must be that $\phi[x, y] \neq 0$. For if it were zero, we would have $\phi[H_i, H_i] = 0$ and this implies that the elements of $S \cap H_i$ are ϕ-extreme points of S. Define $z = [x, y]$. By the Jacobi identity together with formula (1) we find that

$$[t, z] = 2\alpha z + [x, u'] + [y, v'] \tag{2}$$

for some u' and v' in H_{i+1}. It follows that

$$\phi[t, z] = 2\alpha\phi(z) \tag{3}$$

since ϕ vanishes on $[G, H_{i+1}]$. We claim that $z \in H_{i+1}$. To see this, note that $z = [x, y] \in H_i$ so $z = ax + by + w$ for some $w \in H_{i+1}$ and $a, b \in R$. But then

$$[z, x] \quad = \quad b[y, x] + [w, x] = -bz \qquad (\text{mod } H_{i+1})$$

and

$$[z, y] \quad = \quad a[x, y] + [w, y] = az \qquad (\text{mod } H_{i+1}).$$

Since $z \in [G, G]$, $\alpha(z) = \beta(z) = 0$; from (1) it follows that $[z, x]$ and $[z, y]$ are in H_{i+1}. Thus, $-bz$ and az are zero (mod H_{i+1}) which forces $z \in H_{i+1}$ and $a = b = 0$. It follows that $\phi[t, z] = 0$, and by (3), since $\phi(z) \neq 0$ it must be that $\alpha = 0$.

We now have $\alpha = 0$ and $\beta \neq 0$. By replacing t with $T = (1/\beta)t$, we obtain the formulas

$$[T, x] = -y + u \qquad \text{and} \qquad [T, y] = x + v,$$

with $u, v \in H_{i+1}$.

We would like to have $u, v \in J$ for the elements defined in (4), but that need not be the case. So let $X = x + cz$ and $Y = y + dz$ for real numbers c and d to be determined. Now,

$$[T, X] = [T, x + cz] = -y + u + [T, cz] = -Y + dz + u + [T, cz].$$

Using the fact that $z \in H_{i+1}$, so $\phi[T, cz] = 0$, and that $\phi(z) \neq 0$, we can select d so that $\phi(dz + u + [T, cz]) = d\phi(z) + \phi(u) = 0$. Similarly, we select c. Then we have

$$[T, X] = -Y \pmod{J} \qquad \text{and} \qquad \text{[T, Y] = X} \quad \text{(mod J)}.$$

Finally, let $Z = [X, Y] = [x+cz, y+dz]$ and observe that $\phi(Z) = \phi(z)$, since $z \in H_{i+1}$ implies $\phi[x, z] = \phi[y, z] = 0$.

To complete the proof, we note that K is a subalgebra, J is an ideal of K, and K/J is clearly isomorphic to D.

Example. Let G have basis $\{x_1, x_2, x_3, x_4, x_5\}$ over $\mathbb{R}$ and let the non-zero brackets among basis elements be $[x_1, x_2] = x_4, [x_1, x_4] = x_5$, and $[x_2, x_3] = x_5$. Consider the dual element ϕ defined by $\phi(x_1) = \phi(x_2) = \phi(x_3) = \phi(x_4) = 0$ and $\phi(x_5) = 1$. In [5], Vergne points out with this example that the construction given in that paper cannot yield the polarization $H = \mathbb{R}x_1 + \mathbb{R}x_3 + \mathbb{R}x_5$ at ϕ. But the extreme point construction yields H as follows: the radical of ϕ is $\mathbb{R}x_5$ so $S = G \setminus \mathbb{R}x_5$. Choose $y_0 = x_3$; this is a ϕ-extreme point of S because $G(x_3) \cap S = \mathbb{R}x_3 \setminus \{0\}$ which has all zero brackets. Then $G_1 = \mathbb{R}x_1 + \mathbb{R}x_3 + \mathbb{R}x_4 + \mathbb{R}x_5$ and the radical R_1 of $\phi|G_1$ is $\mathbb{R}x_3 + \mathbb{R}x_5$. Let $S_1 = G_1 \setminus R_1$ and choose $y_1 = x_1$. This is a $\phi|G_1$-extreme point of S_1 since $G_1(x_1) \cap S_1 = \mathbb{R}x_1 \setminus \{0\}$ has all zero brackets. Now $G_2 = \{x \in G_1 : \phi[x, z] = 0 \text{ for all } z \in G_1(x_1) \cap S_1\} = \mathbb{R}x_1 + \mathbb{R}x_3 + \mathbb{R}x_5 = H$ and the construction is done.

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