

A New Model for Ribbons in \mathbb{R}^3

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Let's start with a hands-on demonstration. We take a long, rectangular strip of adhesive tape (or postage stamps) and wind it diagonally around a cylinder in such a way that the sticky surface of the tape is entirely in contact with the cylinder. It follows that the two long edges of the tape must lie along parallel helices. This is true by definition— if we start with a line in a plane and then wrap the plane around a cylinder we get a helix. What about the short edges of the tape? Well, by the same definition, they must lie along parallel helices that are orthogonal to the first two! We call the resulting, familiar object a helical band (Figure 1). It plays a central role in this discussion.



Figure 1: A helical band

Here is a more general demonstration. Starting with a long, rectangular strip of paper, we fix in our minds a three-dimensional curve C having the same length as the strip. Now, we force one long edge of the strip (we call the strip a ribbon from now on) to follow the imagined curve. Surely, the whole ribbon must bend and twist in response. Can we describe the shape of the ribbon? More to the point, can we present a mathematical model that would help us visualize such a ribbon and might help us illustrate or understand the curve?

At this point, it is helpful to introduce some terminology. In the plane of the rectangular strip of paper, let's choose an orientation so that the cross-sections parallel to the long edges are horizontal and the cross-sections parallel to the short edges are vertical. When the ribbon is bent along a curve in space, these horizontal and vertical cross-sections are positioned along curves that we call *threads* and *transversals*, respectively. We force one thread (one long edge of the strip) to follow the given curve C . But what can we say about the other threads and about the transversals? Should the other threads be pictured as congruent copies of C ? Should the transversals be pictured as line segments orthogonal to C ? Before trying to answer these questions, it is very important to state that our limited goal is to construct the threads and transversals of a realistic-looking ribbon—we are only hoping for verisimilitude, and make no assertions about the physics of the situation.

As we have already seen, if the curve C is a helix then a helical band can be formed as one physical realization of a ribbon following this curve. All the threads of the ribbon are helices parallel to C , and all the transversals are helices orthogonal to C . But this realization is not unique. For example, without changing the position of one edge along the given helix, an opposite corner can be curled up so that we no longer have a helical band. Thus, we cannot claim that forcing one edge of a ribbon to follow a given curve uniquely determines the shape of the ribbon.

How do we construct a ribbon model? Let's focus on the transversals. The main requirements are that each transversal must be a curve of the proper length and be orthogonal to C , since these properties appear to be true of real-life ribbons. The simplest choice (perhaps too simple) is to use line segments. If each transversal is rendered as a line segment orthogonal to C then we call the model a (generalized) Linear Transversal (LT) model. An LT model may be a reasonable way to thicken a curve, and this can be useful in computer graphics. If a curve has been thickened into some sort of ribbon or tube, then hidden-surface and lighting techniques can be effective in illustrating the curve. However, the LT model often won't provide a

realistic shape. In particular, it fails to yield a helical band when C is a helix (Figure 2). On the other hand, in our new model, we propose to emulate the helical band and render each transversal as an arc of a helix. This is

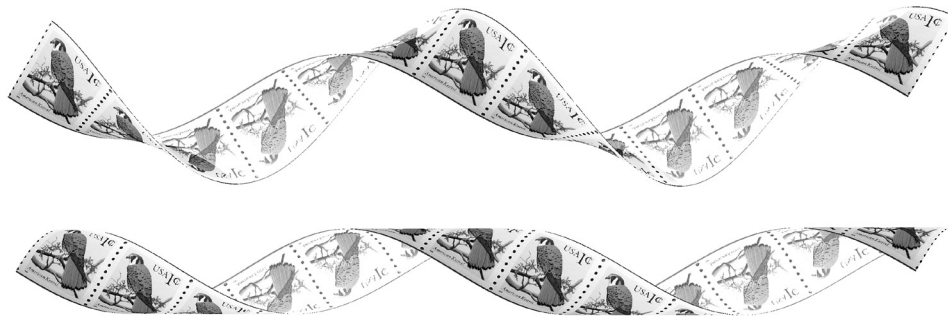


Figure 2: LT model (top) and HT model (bottom)

the Helical Transversal (HT) model. The transversals at the ends of the ribbons in Figure 3 show the subtle, but significant, difference between the two models. Of course, both the LT and HT models are just models and they

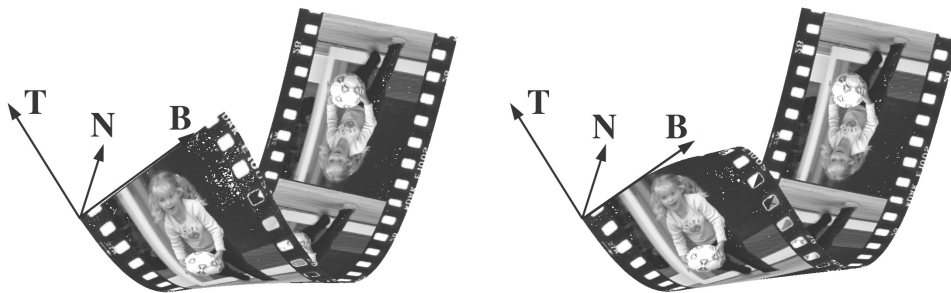


Figure 3: LT model (left) and HT model (right)

do not yield the exact shapes of real-world ribbons, except in special cases. Even so, models can provide plausible and attractive pictures helping to illustrate interesting curves. Indeed, the illustrations in this article have been produced using MATLAB to implement the formulas for both models. By

the way, computer-drawn ribbons are used in science to visualize certain large molecules. For example, Carson [1] has employed a construction of threads (rather than transversals) to form ribbons that are used in understanding and illustrating the molecular structure of proteins. Our focus on transversals is apparently a different approach than the one used by Carson.

For simplicity, throughout this discussion we assume that the curve C can be given in parametric form $s \mapsto \langle x(s), y(s), z(s) \rangle$, with arc length $s \in [0, l]$ as parameter and in terms of functions that are sufficiently differentiable. To be more precise, although we won't dwell on this matter, we assume that the fifth derivatives of the coordinate functions exist and are continuous on $[0, l]$. This suffices to force the curvature and torsion functions (discussed in the next section) to have at least two continuous derivatives. We further assume that C never intersects itself and that it has positive curvature at every point. Such assumptions are commonly used in the study of space curves, and we want to focus on the successful modeling of ribbons rather than on the potential pitfalls of troublesome curves.

The models

For the purposes of this article, we think of a real-world ribbon (or a flexible ruler, a roll of postage stamps, a strip of film, etc.) as being made of a material that cannot be distorted or stretched or shrunk in any direction. A mathematical way to describe such a ribbon is by a ribbon map, defined on a rectangular domain D . Let $D = [0, l] \times [0, w]$ (typically with $l \gg w$), where l is the length and w is the width of the ribbon. Now let's define a *ribbon map* to be a function $\mathbf{f} : D \rightarrow \mathbb{R}^3$, having continuous first partial derivatives on D , and satisfying the following three properties:

(R1) The transversal $v \mapsto \mathbf{f}(s, v)$ has $v \in [0, w]$ as its arclength parameter, for every fixed $s \in [0, l]$. That is, $\mathbf{f}_v \cdot \mathbf{f}_v \equiv 1$.

(R2) The thread $s \mapsto \mathbf{f}(s, v)$ has $s \in [0, l]$ as its arclength parameter, for

every fixed $v \in [0, w]$. That is, $\mathbf{f}_s \cdot \mathbf{f}_s \equiv 1$.

(R3) Transversals and threads intersect at right angles. That is, $\mathbf{f}_s \cdot \mathbf{f}_v \equiv 0$.

Some readers may recognize that these properties amount to \mathbf{f} being a local isometry. Indeed, in terms of the classical notation in the study of surfaces, the properties can be compactly expressed as $G = 1$, $E = 1$, and $F = 0$.

Now here is a more precise version of our problem of forcing one edge of a ribbon to follow a given curve. Let C be a curve of finite length l in \mathbb{R}^3 and let $D = [0, l] \times [0, w]$ be given. We would like to produce a ribbon map $\mathbf{f} : D \rightarrow \mathbb{R}^3$ such that $s \mapsto \mathbf{f}(s, 0)$ is the curve C .

Since the author is not aware of exact solutions to this problem of finding ribbon maps, except in special cases, we shall speak of approximate solutions—models—instead. A *ribbon model* for D and C is a map $\mathbf{f} : D \rightarrow \mathbb{R}^3$ (with continuous first partials on D) such that $s \mapsto \mathbf{f}(s, 0)$ is the curve C , and such that (R1) holds. Whether the map \mathbf{f} satisfies (R2) and (R3) is what separates a ribbon model from an exact ribbon map.

By the way, our definition of a ribbon map does not require the map to be one-to-one. Likewise, a ribbon model may not be one-to-one. This flexibility might be desirable, as in the case of modeling a helical band that partially overlaps itself. Alternatively, the domain of a ribbon map that is not one-to-one can be restricted to a subset $D_1 = [0, l] \times [0, w_1]$ of D in order to achieve a one-to-one result. In effect, we would just be choosing a narrower ribbon that is easier to bend along a curve without bumping into itself.

Again, let C be represented by the arclength parameterization $s \mapsto \langle x(s), y(s), z(s) \rangle$. Ribbon models are expressed in terms of the *Frenet frame*, a 'moving' coordinate system that we construct at each point of the curve. At the point $P = P(s)$ on the curve, we define unit vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} . The *unit tangent vector* \mathbf{T} is a vector of length 1 pointing in the direction of the derivative $\langle x'(s), y'(s), z'(s) \rangle$. The *unit normal vector* \mathbf{N} is the vector of length 1 in the direction of $\frac{d\mathbf{T}}{ds}$. We can understand \mathbf{T} and \mathbf{N} by thinking of an object moving at unit speed along the curve. In this situation, $\mathbf{T}(s)$ is

the velocity vector at time s and $\mathbf{N}(s)$ is in the direction of the acceleration vector. These vectors are necessarily orthogonal to each other because

$$0 = \frac{d}{ds}(1) = \frac{d(\mathbf{T} \cdot \mathbf{T})}{ds} = 2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds}.$$

We complete the coordinate system (the Frenet frame) by defining the *unit binormal vector* \mathbf{B} to be $\mathbf{T} \times \mathbf{N}$. The properties of cross products guarantee that this is a unit vector orthogonal to both \mathbf{T} and \mathbf{N} .

Our discussion of ribbon models also requires the notions of curvature and torsion along a curve. These are scalar quantities that characterize the geometry of the curve. Using the Frenet frame, we have a straightforward way to define and describe the *curvature* κ and *torsion* τ of a space curve. In fact, the *Frenet-Serret* formulas (see [4] or [3]) tell the story:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}. \quad (1)$$

By the first of these formulas, the curvature κ is a measure of how the curve is turning within the *osculating plane* spanned by \mathbf{T} and \mathbf{N} . And by the third formula, since \mathbf{B} is normal to the osculating plane, the torsion τ is a measure of how the osculating plane is turning—or how the curve is twisting.

As one example, the torsion of a circle, or of any other plane curve, is zero since the osculating plane is fixed. As a second example, think of a helix. We derive in the next section the fact that both the curvature and the torsion of a helix are constants. A nonzero, constant torsion reflects the understanding that, along a helix, the vector \mathbf{B} turns at a constant rate.

In the Linear Transversal model, each transversal is the line segment of length w starting at a point P and heading in the direction of \mathbf{B} . Why \mathbf{B} ? The easiest way to explain this choice is to think of the special case in which the given curve is planar. In this case, the ribbon can be naturally realized as a (right, noncircular) cylinder. For example, imagine a coil of postage stamps standing on a desktop and following a spiral curve in the plane of the desktop (Figure 4). The vectors \mathbf{T} and \mathbf{N} are in the plane of the curve and the transversals are line segments in the direction of \mathbf{B} or $-\mathbf{B}$. For simplicity,

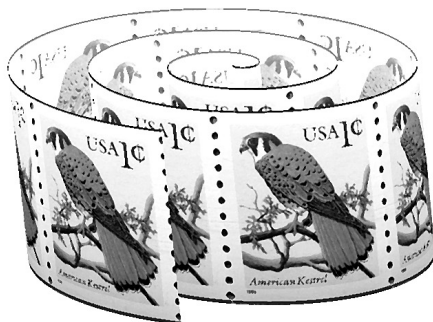


Figure 4: Ribbon for a planar curve

we focus first on one choice of sign. The formula for the LT model is simply

$$\mathbf{f}(s, v) = \langle x(s), y(s), z(s) \rangle + v\mathbf{B}(s), \quad (2)$$

where $(s, v) \in D$. The differentiability assumptions for the curve guarantee that the first partials of \mathbf{f} exist and are continuous on D . Also, it is clear by our construction of \mathbf{f} that (R1) holds. In fact, $\mathbf{f}_v \cdot \mathbf{f}_v = \mathbf{B} \cdot \mathbf{B} \equiv 1$. Thus, the LT model is a ribbon model.

Similar models are obtained by replacing \mathbf{B} in this formula with another unit vector in the normal plane to the curve. In this way, we could regard the *generalized* LT model to be

$$\mathbf{f}(s, v) = \langle x(s), y(s), z(s) \rangle + v((\cos q)\mathbf{N} + (\sin q)\mathbf{B}), \quad (3)$$

where $s \mapsto q(s)$ is a continuously differentiable function. We investigate these models in the next section.

The Helical Transversal model is strongly motivated by the helical band. We have already noted that when a rectangle is wrapped around a (right, circular) cylinder, the boundaries of the rectangle become helical arcs on the cylinder. The wrapping operation we speak of preserves distances and angles, so it sends perpendicular lines to orthogonal helices. Thus, in fact, for any helix H through a point P on a given cylinder, there is a unique helix H^{orth} lying on the same cylinder and orthogonal to H at P .

There are two fundamental ideas behind the HT model. One is that we may view a curve as being approximated at each point by a certain helix—in general, a different helix at each point. In fact, we show in Theorem 1 that at any point P of a given curve C there is a unique, approximating helix that has the same curvature, torsion and Frenet frame at P . This is the main result in a MAGAZINE article by McHugh [2]. The other fundamental idea is to choose the transversal at P to be an arc of the helix that is orthogonal to the approximating helix, just as if the ribbon were a helical band. In order to work out the formula for this HT model, we need to do some preparation in the next section.

By the way, a quick comparison of the two illustrations in Figure 5 shows that the LT and HT models can appear remarkably similar. In this case, however, the HT model seems to do a better job of portraying a ribbon whose bottom edge follows a 3-dimensional, spiral curve C . (Look at the "tightly wound" end of the spiral.)

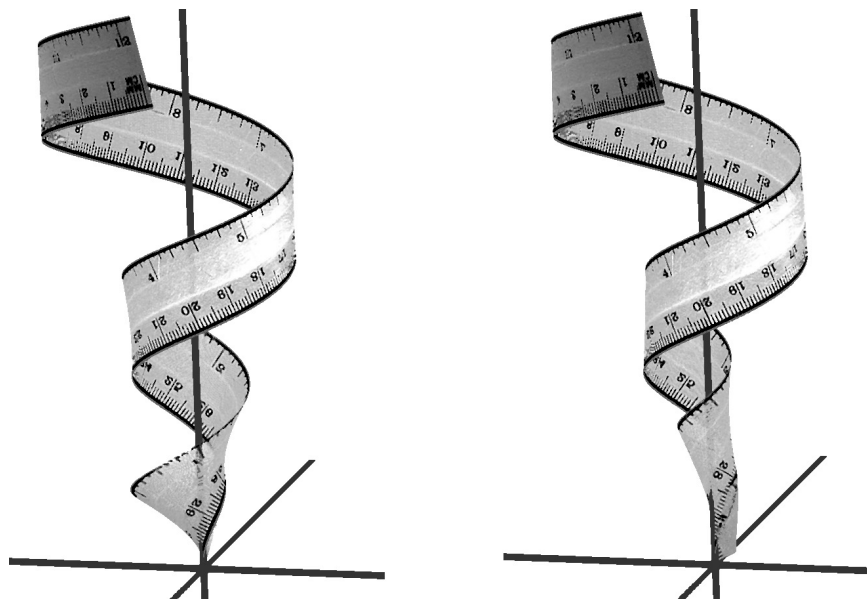


Figure 5: LT model (left) and HT model (right)

Helix redux

The helix plays a major role in the study of space curves. Just as the shape of a plane curve at a point is compared to a circle by calculating the curvature, the shape of a space curve at a point can be compared to a helix by calculating the curvature and the torsion.

Tangent vectors to a helix make a constant angle $\theta \in [0, \pi]$ with the chosen direction vector of the axis of the cylinder. Note that $\theta = \pi/2$ corresponds to a circle and $\theta = 0$ or π corresponds to a line parallel to the axis of the cylinder. These are degenerate helices and we will avoid them in the rest of this section.

A helix on a given cylinder can be right-handed, like the threads of a standard screw, or left-handed. One way to characterize the difference is in terms of the tangent vector \mathbf{T} and the binormal vector \mathbf{B} at each point. Once the positive direction of the axis of the cylinder is selected, a right-handed helix has the property that the angles that \mathbf{T} and \mathbf{B} make with the axis are both acute or both obtuse. By contrast, for a left-handed helix one of these angles is acute and the other is obtuse.

We need formulas describing the geometry of a helix and, by choosing convenient coordinates, it is enough to work with a helix on the cylinder of radius r having the z -axis as its directed axis and passing through the point $(r, 0, 0)$. Such a helix can be parameterized by

$$\begin{cases} x = r \cos t \\ y = r \sin t \\ z = ct \end{cases} \quad t \in \mathbb{R} \quad (4)$$

The helix is right-handed if $c > 0$ and left-handed if $c < 0$.

Let s denote an arclength parameter for the helix (4), so

$$\frac{ds}{dt} = \left| \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \right| = \sqrt{r^2 + c^2}.$$

It is routine to calculate the Frenet frame for this helix. The vectors are

$$\mathbf{T} = \langle -r \sin t, r \cos t, c \rangle / \sqrt{r^2 + c^2},$$

$$\mathbf{N} = \frac{d\mathbf{T}}{dt} / \left| \frac{d\mathbf{T}}{dt} \right| = \langle -\cos t, -\sin t, 0 \rangle$$

and

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \langle c \sin t, -c \cos t, r \rangle / \sqrt{r^2 + c^2}.$$

Next we calculate the curvature and torsion of the helix (4). The curvature is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{r}{r^2 + c^2}. \quad (5)$$

And, since

$$\frac{d\mathbf{B}}{ds} = \langle c \cos t, c \sin t, 0 \rangle / (r^2 + c^2) = -\frac{c}{(r^2 + c^2)} \mathbf{N} = -\tau \mathbf{N},$$

it is apparent that the torsion is

$$\tau = \frac{c}{r^2 + c^2}. \quad (6)$$

It is worth noting that both the curvature and the torsion are constant. This is a property that characterizes helices, in the sense that a space curve is a (circular) helix if and only if the curvature and torsion are constant.

Observe that the angle θ between the tangent vector \mathbf{T} of the helix and the positive z -axis satisfies

$$\cos \theta = \mathbf{T} \cdot \mathbf{k} = \frac{c}{\sqrt{r^2 + c^2}}. \quad (7)$$

It follows that

$$\sin \theta = \frac{r}{\sqrt{r^2 + c^2}} = \mathbf{B} \cdot \mathbf{k}, \quad (8)$$

since $\sin \theta$ is positive on the interval $(0, \pi)$. Finally, note that the arclength along the helix between the two points $(r, 0, 0)$ and $(r \cos t, r \sin t, ct)$ is given by

$$\int_0^t \frac{ds}{dt} dt = \sqrt{r^2 + c^2} |t|. \quad (9)$$

We actually apply this arclength formula to the orthogonal helix. For the helix (4), the orthogonal helix drawn on the same cylinder and passing through the point $(r, 0, 0)$ is given by

$$\begin{cases} x = r \cos t \\ y = r \sin t \\ z = \frac{-r^2}{c}t \end{cases} \quad t \in \mathbb{R} \quad (10)$$

To see that these two helices ((4) and (10)) are orthogonal at the point $(r, 0, 0)$, just note that their tangent vectors have dot product equal to zero.

Developing the HT model

Now that we have the necessary background, we can explain the HT model. **THEOREM 1.** Assume we are given the following data: a point $P \in \mathbb{R}^3$, orthogonal unit vectors \mathbf{T} , \mathbf{N} , and $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, and real numbers $\kappa > 0$ and τ . Then there is exactly one helix passing through P and having κ as curvature, τ as torsion, and \mathbf{T} , \mathbf{N} , and \mathbf{B} as Frenet frame at P .

Proof. The formulas (5) and (6) for the curvature and torsion of a helix allow us to uniquely choose the radius r of the needed cylinder, and to calculate the needed parameter c . In fact,

$$r = \frac{\kappa}{\kappa^2 + \tau^2} \quad \text{and} \quad c = \frac{\tau}{\kappa^2 + \tau^2}.$$

Also, from (7) and (8) we find that the angle θ between the tangent vector \mathbf{T} of our intended helix at P and the unknown axis of the cylinder (with a suitable direction vector \mathbf{A}) satisfies

$$\cos \theta = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \quad \text{and} \quad \sin \theta = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}.$$

To locate the axis of the cylinder, note that the normal vector at any point of a helix is directed towards the axis and is orthogonal to \mathbf{A} . Thus, the axis we seek passes through a point that is a distance r from the point P in the direction of \mathbf{N} and satisfies $\mathbf{A} \cdot \mathbf{N} = 0$. Furthermore, (7) and (8) require

$\mathbf{T} \cdot \mathbf{A} = \cos \theta$ and $\mathbf{B} \cdot \mathbf{A} = \sin \theta$. It follows that $\mathbf{A} = (\cos \theta)\mathbf{T} + (\sin \theta)\mathbf{B}$, which is uniquely determined by the data. The axis, then, passes through the terminal point Q of the position vector $\vec{OP} + r\mathbf{N}$, and has direction vector \mathbf{A} .

To present a formula for the desired helix, we use the coordinate system with Q as origin and with direction vectors $-\mathbf{N}$, $\mathbf{A} \times (-\mathbf{N})$, and \mathbf{A} . These are the appropriate vectors because \mathbf{A} is the direction of the axis of the cylinder (playing the role of the z -axis), and $-\mathbf{N}$ points in the direction of \vec{QP} (playing the role of the x -axis). The other vector, $\mathbf{A} \times (-\mathbf{N})$, is exactly the one that completes the right-handed system. Now a point $X = (x(t), y(t), z(t))$ of the helix is determined, using the vector form of (4), by

$$Q\vec{X} = (r \cos t)(-\mathbf{N}) + (r \sin t)(\mathbf{A} \times (-\mathbf{N})) + (ct)(\mathbf{A}).$$

It is not difficult to verify that this helix has the stated properties. \square

The construction used in the proof of the theorem is the first step in deriving a formula for the HT ribbon model. Assume that a curve C is given in parametric form by $P(s) = (x(s), y(s), z(s))$, where $0 \leq s \leq l$ and s is arclength. Also, let $D = [0, l] \times [0, w]$ be given. According to Theorem 1, at each point $P(s)$ the curve has the same curvature, torsion, and Frenet frame as a certain, unique helix (which varies with s). As in the proof of the theorem, this helix is on a cylinder whose axis passes through the terminal point Q of the position vector $\vec{OP} + r\mathbf{N}$, and has direction vector $\mathbf{A} = (\cos \theta)\mathbf{T} + (\sin \theta)\mathbf{B}$. Of course, all the quantities involved in this discussion are functions of s , since they vary along the curve.

Now we need to construct the orthogonal helix at P on the same cylinder. As in the proof of the theorem, we locate the origin at Q and use the vectors $-\mathbf{N}$, $\mathbf{A} \times (-\mathbf{N})$, and \mathbf{A} as the coordinate directions. In these terms and using (10), a point X of the orthogonal helix satisfies

$$Q\vec{X} = (r \cos t)(-\mathbf{N}) + (r \sin t)(\mathbf{A} \times (-\mathbf{N})) + \left(\frac{-r^2}{c}t\right)(\mathbf{A}), \quad (11)$$

where

$$r = \frac{\kappa}{\kappa^2 + \tau^2}, \quad c = \frac{\tau}{\kappa^2 + \tau^2}, \quad (12)$$

and t is a parameter. Of course, r and c , like κ and τ , are functions of s . This comment is to emphasize that the specific orthogonal helix emanating from the point $P(s)$ varies with s . The shape of the orthogonal helix is governed by r and c , and these parameters, in turn, are determined by the curvature and torsion of the curve at $P(s)$.

The HT model $\mathbf{f} : D \rightarrow \mathbb{R}^3$ is characterized by requiring each transversal $v \mapsto \mathbf{f}(s, v)$ to be an arc of the orthogonal helix starting at $P(s)$ and having v as arclength parameter. Thus, each point $(s, v) \in D$ is sent to the point X as in (11), chosen so that the arclength from $P(s)$ to X is equal to v . According to (9), this says we take the parameter t in (11) to satisfy

$$|t| = \frac{v}{\sqrt{r^2 + (-r^2/c)^2}} = \frac{v|c|}{r\sqrt{r^2 + c^2}}.$$

We pick

$$t = \frac{-vc}{r\sqrt{r^2 + c^2}} \quad (13)$$

from among the two options. It turns out that with this choice the resulting formula for the HT model is in agreement with the LT model, when the curve happens to be planar. Finally (see Figure 6), since $\vec{OX} = \vec{OP} - \vec{QP} + \vec{QX} = \vec{OP} + r\mathbf{N} + \vec{QX}$, we can revise (11) and write an explicit formula for the HT ribbon model:

$$\begin{aligned} \mathbf{f}(s, v) &= \langle x(s), y(s), z(s) \rangle \\ &+ r\mathbf{N} + (r \cos t)(-\mathbf{N}) + (r \sin t)(\mathbf{A} \times (-\mathbf{N})) + \left(\frac{-r^2}{c}t\right)(\mathbf{A}). \end{aligned}$$

If we use $\mathbf{A} = (\cos \theta)\mathbf{T} + (\sin \theta)\mathbf{B}$ and simplify, we obtain

$$\begin{aligned} \mathbf{f}(s, v) &= \langle x(s), y(s), z(s) \rangle \\ &+ [r \sin t \sin \theta - \frac{r^2}{c}t \cos \theta]\mathbf{T} + [r - r \cos t]\mathbf{N} + [-r \sin t \cos \theta - \frac{r^2}{c}t \sin \theta]\mathbf{B}. \end{aligned}$$

Next, we replace r , c , $\cos \theta$, $\sin \theta$, and t with their equivalent formulas ((12), (7), (8), (13)) involving κ and τ . The result is

$$\mathbf{f}(s, v) = \langle x(s), y(s), z(s) \rangle$$

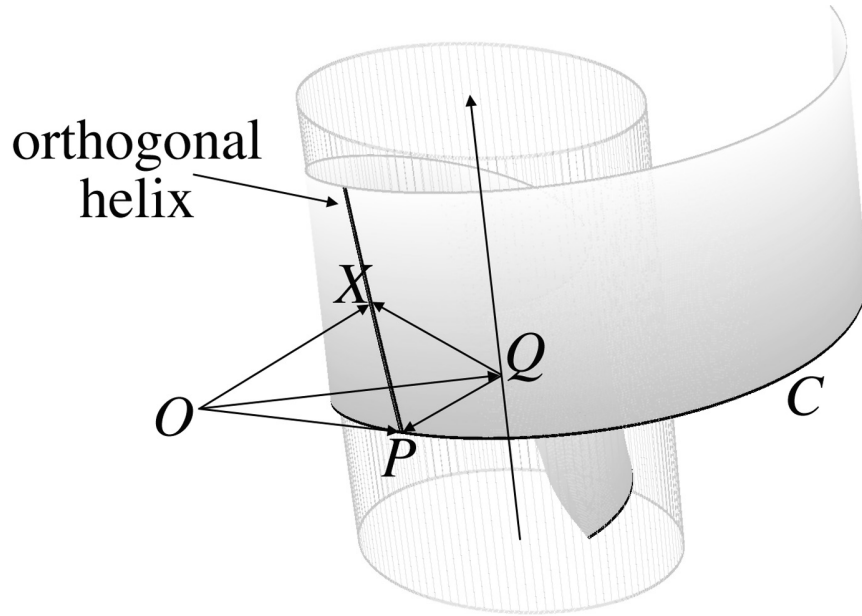


Figure 6: Constructing transversals in the HT model

$$\begin{aligned}
& + \frac{\kappa}{\kappa^2 + \tau^2} \left\{ \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \sin\left(\frac{-v\tau\sqrt{\kappa^2 + \tau^2}}{\kappa}\right) + v\tau \right] \mathbf{T} \right. \\
& + \left(1 - \cos\left(\frac{-v\tau\sqrt{\kappa^2 + \tau^2}}{\kappa}\right) \right) \mathbf{N} \\
& \left. + \left[\frac{-\tau}{\sqrt{\kappa^2 + \tau^2}} \sin\left(\frac{-v\tau\sqrt{\kappa^2 + \tau^2}}{\kappa}\right) + v\kappa \right] \mathbf{B} \right\}.
\end{aligned}$$

Although this expression prominently shows the dependence on κ and τ , we can simplify the appearance of the formula by letting $\rho = \sqrt{\kappa^2 + \tau^2}$ and $t = -v\tau\rho/\kappa$ to obtain what we will refer to as the HT model:

$$\begin{aligned}
\mathbf{f}(s, v) = \langle x(s), y(s), z(s) \rangle + \frac{\kappa}{\rho^2} \left\{ \left[\frac{\kappa}{\rho} \sin t + v\tau \right] \mathbf{T} \right. \\
\left. + (1 - \cos t) \mathbf{N} + \left[\frac{-\tau}{\rho} \sin t + v\kappa \right] \mathbf{B} \right\}. \tag{14}
\end{aligned}$$

It is not hard to see from this formula that the first partials of \mathbf{f} exist and are continuous on D . And the construction guarantees that (R1) holds.

Thus, the HT model is, indeed, a ribbon model. We explore this matter further in the next section. For now, it is revealing to note that there is a component in the tangent direction—the transversal is not just drawn in the normal plane at each point of the curve as is the case with all of the LT models. We can also see in this formula, that if $\tau \rightarrow 0$ then $t \rightarrow 0$ and, in the limit, we obtain the LT model (2).

When do the LT and HT models become ribbon maps?

We need to test each model to determine if the ribbon map properties (R2) and (R3) hold. Let's look first at (R2), the property saying that distance is preserved along threads. This is equivalent to requiring that, for each fixed v , the magnitude of $\mathbf{f}_s(s, v)$ is identically equal to 1.

In the case of the LT model, we have

$$\mathbf{f}_s(s, v) = \frac{d}{ds} \{ \langle x(s), y(s), z(s) \rangle + v\mathbf{B}(s) \} = \mathbf{T}(s) - v\tau\mathbf{N}(s),$$

and it follows that

$$\mathbf{f}_s \cdot \mathbf{f}_s = 1 + v^2\tau^2. \tag{15}$$

Thus, (R2) fails to hold in the case of the LT model, since $\mathbf{f}_s \cdot \mathbf{f}_s > 1$ (if $v \neq 0$) unless the torsion τ of the curve is zero at every point. This is the same as saying that the curve C must be planar.

On the other hand, the LT model satisfies (R3) for every curve. That is, the transversals are orthogonal to the threads at every point. To see this, we calculate the dot product

$$\mathbf{f}_s \cdot \mathbf{f}_v = (\mathbf{T}(s) - v\tau\mathbf{N}(s)) \cdot \mathbf{B}(s) = 0.$$

The generalized LT models (3) behave in a similar way. In fact, (R2) fails unless $\tau = 0$ but (R3) always holds. For if

$$\mathbf{f}(s, v) = \langle x(s), y(s), z(s) \rangle + v((\cos q)\mathbf{N} + (\sin q)\mathbf{B})$$

then, by the Frenet-Serret formulas (1),

$$\begin{aligned}\mathbf{f}_s(s, v) &= \mathbf{T} + v((- \sin q)q'\mathbf{N} + \cos q(-\kappa\mathbf{T} + \tau\mathbf{B}) + (\cos q)q'\mathbf{B} + (\sin q)(-\tau\mathbf{N})) \\ &= (1 - v\kappa \cos q)\mathbf{T} - v(q' + \tau)(\sin q)\mathbf{N} + v(q' + \tau)(\cos q)\mathbf{B}\end{aligned}$$

and

$$\mathbf{f}_v(s, v) = (\cos q)\mathbf{N} + (\sin q)\mathbf{B}.$$

Thus,

$$\mathbf{f}_s \cdot \mathbf{f}_v = -v(q' + \tau) \sin q \cos q + v(q' + \tau) \cos q \sin q = 0,$$

so (R3) holds. Also, after a little simplification,

$$\mathbf{f}_s \cdot \mathbf{f}_s = (1 - v\kappa \cos q)^2 + v^2(q' + \tau)^2.$$

The only way to make this expression be identically equal to 1 is to have $\cos q = 0$ and $q' + \tau = 0$. But $\cos q = 0$ means that the model is reduced to one of the special LT models in which the transversals are drawn in the direction of $\pm\mathbf{B}$. It follows that q is constant, $q' = 0$, and $\tau = 0$. That is, the curve must once again be planar for (R2) to hold. The conclusion is that the LT model and all the generalized LT models are ribbon maps if and only if the given curve is planar.

Before we begin the discussion of whether the HT function (14) satisfies the requirements to be a ribbon map, it may be useful to make an additional remark for the purposes of comparison. It seems reasonable to say that the LT model is designed for planar curves and can be expected to provide a good approximation to a ribbon map if τ is close to zero (so the curve is close to being planar). Indeed, this is born out by our computation of $\mathbf{f}_s \cdot \mathbf{f}_s$ as in (15). By contrast, the HT model is designed for helices (κ and τ constant) and should be expected to provide a good approximation to a ribbon map if κ' and τ' are close to zero. The next theorem verifies that this is the case.

We need to analyze the partial derivatives of the HT ribbon model \mathbf{f} given by (14), and straightforward (though somewhat arduous) computations yield the expressions below.

One partial derivative is

$$\mathbf{f}_v(s, v) = \delta_1 \mathbf{T} + \delta_2 \mathbf{N} + \delta_3 \mathbf{B},$$

where

$$\delta_1 = \frac{\kappa\tau(1 - \cos t)}{\rho^2}, \quad \delta_2 = \frac{-\tau \sin t}{\rho}, \quad \text{and} \quad \delta_3 = \frac{\tau^2 \cos t + \kappa^2}{\rho^2}. \quad (16)$$

Just as in (14), $\rho = \sqrt{\kappa^2 + \tau^2}$ and $t = -v\tau\rho/\kappa$.

The other partial derivative is

$$\mathbf{f}_s(s, v) = (\alpha_1\kappa' + \beta_1\tau' + \gamma_1)\mathbf{T} + (\alpha_2\kappa' + \beta_2\tau' + \gamma_2)\mathbf{N} + (\alpha_3\kappa' + \beta_3\tau' + \gamma_3)\mathbf{B},$$

where

$$\gamma_1 = 1 + \frac{\kappa^2(\cos t - 1)}{\rho^2}, \quad \gamma_2 = \frac{\kappa \sin t}{\rho}, \quad \gamma_3 = \frac{\kappa\tau(1 - \cos t)}{\rho^2}, \quad (17)$$

and the alphas and betas are somewhat more complicated functions of the same sorts. They are sums of fractions whose denominators are powers of ρ and whose numerators are polynomials in κ , τ , v , $\sin t$, and $\cos t$. For example,

$$\alpha_1 = \frac{(2\kappa\tau^2 - \kappa^3) \sin t}{\rho^5} + \frac{v(\tau^3 - \kappa^2\tau + \tau^3 \cos t)}{\rho^4}.$$

It is clear from the form of all these *coefficient functions* (the alphas, betas, gammas, and deltas), that they are continuous functions on D . This is because κ , and hence ρ , is nonzero for $(s, v) \in D$. It follows that the coefficient functions are bounded on the compact set D . To shorten the statement of the following theorem, we can take advantage of the vector notation $\alpha = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ and similarly for β , γ , and δ .

THEOREM 2. Let C be a curve of length l in \mathbb{R}^3 and let $D = [0, l] \times [0, w]$. The HT function $\mathbf{f} : D \rightarrow \mathbb{R}^3$ given by (14) satisfies the following properties.

(i) Distance is preserved along transversals (i.e. (R1) holds):

$$\mathbf{f}_v \cdot \mathbf{f}_v = \delta_1^2 + \delta_2^2 + \delta_3^2 = \|\delta\|^2 = 1.$$

(ii) The ribbon map condition (R2) holds if $\kappa' = 0 = \tau'$ and it holds approximately if κ' and τ' are sufficiently small:

$$\begin{aligned} \mathbf{f}_s \cdot \mathbf{f}_s &= 1 + \|\alpha\|^2(\kappa')^2 + (2\alpha \cdot \beta)\kappa'\tau' \\ &\quad + \|\beta\|^2(\tau')^2 + (2\gamma \cdot \alpha)\kappa' + (2\gamma \cdot \beta)\tau', \end{aligned}$$

since

$$\|\gamma\| = 1.$$

(iii) The ribbon map condition (R3) holds if $\kappa' = 0 = \tau'$ and it holds approximately if κ' and τ' are sufficiently small:

$$\mathbf{f}_s \cdot \mathbf{f}_v = (\alpha \cdot \delta)\kappa' + (\beta \cdot \delta)\tau',$$

since $\gamma \cdot \delta = 0$.

Proof. We just use the formulas for the partial derivatives, \mathbf{f}_v and \mathbf{f}_s . It is not difficult to verify, using (16) and (17), that $\|\delta\|^2 = 1$, $\|\gamma\| = 1$, and $\gamma \cdot \delta = 0$.

It should be recognized that we are not apt to apply Theorem 2 by letting κ' and τ' tend to zero. Since the curve C would typically be given as part of the data, the functions κ and τ , along with their derivatives, are already determined and fixed. Qualitatively, the theorem just confirms that the HT model is apt to perform best when the curvature and torsion of the curve are changing gradually.

In closing, we return to the empirical aspect of this topic—after all, the HT and LT models actually do provide pictures. What may be most striking about the side-by-side pictures obtained from the two models is how similar they are. Given the relatively complicated derivation of the HT model, we may have expected more dramatic differences between the two. The fact is, however, that the transversals in both models start out in the direction of the same binormal vector \mathbf{B} . In cases where the models are noticeably different, it is an indication of the effect of torsion in the given curve. As a final example, Figure 7 is offered as a suggestion of how a curve can be illustrated using a ribbon to provide some added depth and interest. The curve in this case is given by

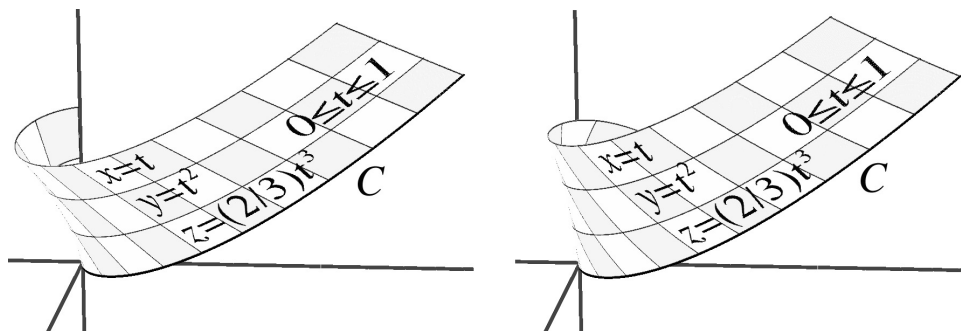


Figure 7: LT model (left) and HT model (right)

$$\begin{cases} x = t \\ y = t^2 \\ z = (2/3)t^3 \end{cases} \quad t \in [0, 1], \quad (18)$$

and has the nice property that the torsion and curvature are the same. In fact,

$$\tau = \frac{2}{(1 + 2t^2)^2} = \kappa.$$

Apparently, the torsion decreases by a factor of 9, from $\tau = 2$ at one endpoint to $\tau = 2/9$ at the other. And perhaps the HT model does a better job of signaling this change. Recall that in the HT model each transversal is plotted along a different helix. And each helix is on a cylinder of varying radius, given in this example by $r = \kappa/(\kappa^2 + \tau^2) = 1/(2\tau)$. The relatively small radius when $t = 0$, one ninth of the radius when $t = 1$, causes the distinctive curl in the HT ribbon. This may not be a dramatic improvement over the result of the LT model, but the curled edge is eye-catching and directs our attention to the high torsion part of the curve.

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