

The principle (+) and club guessing sequences

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Club guessing sequences

Let S be a stationary subset of an uncountable regular cardinal κ consisting of limit ordinals.

Definition A sequence $\vec{C} = \langle C_\gamma : \gamma \in S \rangle$ is called a *tail club guessing (TCG) sequence on S* iff

- (i) each C_γ is an unbounded subset of γ , and
- (ii) for every club subset D of κ , there exists a $\delta \in S$ such that $C_\delta \subseteq^* D$.

If \vec{C} satisfies (i) and (ii) with $C_\gamma \subseteq^* D$ replaced by $C_\gamma \subseteq D$, then it is called a *fully club guessing (FCG) sequence*.

Fact There exists a TCG-sequence on S iff there exists an FCG-sequence on S .

Kunen's Axiom

Definition *Kunen's Axiom (KA)* asserts that there exists a ladder system $\langle L_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ such that for every club subset D of ω_1 , there exists an $\alpha \in \omega_1 \cap \text{Lim}$ such that for all but finitely many $n < \omega$, $D \cap [L_\alpha(n), L_\alpha(n+1)) \neq \emptyset$. Here, $L_\alpha(n)$ denotes the n -th element of L_α .

Fact If there exists a TCG-sequence on ω_1 , then KA holds.

Question (P. Nyikos) Is it consistent that there is no TCG sequence on ω_1 but KA holds?

Attempt

It is easy to see that the forcing to add a bottom to any Q -point will force KA .

So, we are done if it is consistent that

- there is no TCG-sequence on ω_1 ,
- there is a forcing notion that adds a bottom to a Q -point, and
- the forcing adds no new TCG-sequence on ω_1 .

The investigation on $(+)_{<\omega}$ provides an elegant construction of such a model.

The Principle $(+)$ and $(+)_{<\omega}$

The following principles were proposed by P. Larson.

Definition $(+)$ is the principle that asserts the existence of a stationary subset S of $[H(\aleph_2)]^{\aleph_0}$ such that for every $M, N \in S$ with $M \cap \omega_1 = N \cap \omega_1$, for every pair of club subsets $D \in M$ and $E \in N$ of ω_1 we have $D \cap E \cap M \neq \emptyset$.

$(+)_{<\omega}$ is the version of $(+)$ that allows finitely many models. Namely,

Definition $(+)_{<\omega}$ is the principle that asserts the existence of a stationary subset S of $[H(\aleph_2)]^{\aleph_0}$ such that for every $N_1, \dots, N_n \in S$, if $N_i \cap \omega_1 = N_0 \cap \omega_1$ for every i , whenever $D_i \in N_i$ is a club subset of ω_1 for each i , we have $\bigcap_i D_i \cap M \neq \emptyset$.

It is easy to see that the existence of a TCG-sequence implies $(+)_{<\omega}$.

Is $\neg(+)$ consistent?

MRP

Definition MRP is the assertion that for some uncountable set X and uncountable regular cardinal θ with $X \in H(\theta)$, whenever Σ is a mapping from a club subset of $[H(\theta)]^{\aleph_0}$ such that for every $M \in \text{dom}(\Sigma)$,

- $X \in M$ and $\Sigma(M)$ is a subset of $[X]^{\aleph_0}$,
- for every club subset $D \in M$ of $[X]^{\aleph_0}$, $D \cap \Sigma(M) \neq \emptyset$ (M -stationary), and
- for every $Y \in \Sigma(M)$, there exists an y such that $\{Z \in [X]^{\aleph_0} : y \in Z \subseteq Y\} \subseteq \Sigma(M)$ (open),

there exists a continuous \in -chain $\langle N_\alpha : \alpha < \omega_1 \rangle$ in $[H(\theta)]^{\aleph_0}$ such that for every non-zero limit $\delta < \omega_1$, there exists a $\zeta < \delta$ such that for every $\gamma \in [\zeta, \delta)$, $N_\gamma \cap X \in \Sigma(N_\delta)$.

MRP(Cont.)

Facts

- PFA implies MRP.
- MRP implies $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.
- MRP implies there is no club guessing sequence on ω_1 .

Theorem (J. Moore) MRP implies $\neg(+)$.

Proof

- Suppose both MRP and $(+)$ hold. Let T witness $(+)$.
- We may assume that for every $N \in [H(\aleph_2)]^{\aleph_0} \setminus T$, there exist $M_N \in T$ and club subsets $D_N \in M_N$ and $E_N \in N$ of ω_1 such that $M_N \cap \omega_1 = N \cap \omega_1$ and $D_N \cap E_N \cap N = \emptyset$.

Proof(Cont.)

- Recall that MRP implies there is no club guessing sequence on ω_1 .
- So, for every $N \in T$, there exist $M_N \in T$ with $M_N \cap \omega_1 = N \cap \omega_1$ and a club subset $D_N \in M_N$ of ω_1 such that for every club subset $D \in N$ of ω_1 , $D \cap N \not\subseteq^* D_N$.
- Note that in either case, D_N is a club subset of ω_1 so that $D_N \in M_N \in T$.
- For every typical $N \in [H(\theta)]^{\aleph_0}$, let $\Sigma(N) = \{X \in [H(\aleph_2)]^{\aleph_0} : X \cap \omega_1 \notin D_N\}$

Proof(Cont.)

- Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ be a reflecting sequence. Let $E = \{\alpha < \omega_1 : N_\alpha \cap \omega_1 = \alpha\}$.
- Pick an $N \in T$ so that $E \in N$ and let $\delta = N \cap \omega_1$.
- We have $D_{N_\delta} \cap E$ is bounded while $E \in N \in T$ and $D_{N_\delta} \in M_{N_\delta} \in T$ for some M . Contradiction.

$$(+)^p$$

We need the following strengthening of $(+)$.

Definition Let S be a stationary subset of ω_1 . A stationary subset T of $[H(\aleph_2)]^{\aleph_0}$ is *projective stationary on S* iff for every club subset E of $[H(\aleph_2)]^{\aleph_0}$, $\{N \cap \omega_1 : N \in E \cap T\}$ contains a club subset of S .

Definition $(+)^p(S)$ asserts that $(+)$ is witnessed by a projective stationary set on S .

Similarly for $(+)^p_{<\omega}(S)$.

$(+)^p$ is witnessed by club

Theorem (J. Moore) $(+)^p(S)$ implies that there exists a club subset E of $[H(\aleph_2)]^{\aleph_0}$ such that for every $M, N \in E$ with $M \cap \omega_1 = N \cap \omega_1 \in S$, for every pair of club subsets $D \in M$ and $D' \in N$ of ω_1 , $D \cap D' \cap M \neq \emptyset$. Similarly for $(+)_{<\omega}^p(S)$.

Remark If NS_{ω_1} is saturated, then every stationary subset of $[H(\aleph_2)]^{\aleph_0}$ is projective stationary on some stationary subset of ω_1 . Thus, in this case,

- $(+)$ implies $(+)^p(S)$ for some stationary subset S of ω_1 .
- $(+)_{<\omega}$ implies $(+)_{<\omega}^p(S)$ for some stationary subset S of ω_1 .

No ccc forcing adds a CG-sequence

Theorem(I and P. Larson) Assume $\neg(+)<_{\omega}$ and NS_{ω_1} is saturated. Then, no ccc forcing adds a TCG-sequence.

Proof

- Suppose that P is a ccc forcing notion that adds a TCG-sequence. Let $V[G]$ be the extension.
- Since there is a TCG-sequence in $V[G]$, $(+)<_{\omega}$ holds there.
- Since P is ccc, NS_{ω_1} remains saturated in $V[G]$.
- So, $(+)^p_{<_{\omega}}(S)$ holds for some S in $V[G]$. Let E be the club witnessing it.
- Since P is ccc, we may assume $E \in V$.
- Contradiction to the assumption that $(+)<_{\omega}$ fails in V .

Con(KA + \nexists TCG)

Theorem (I and P. Larson) It is consistent relative to the existence of a supercompact cardinal that KA holds but there is no TCG-sequence on ω_1 .

Proof

- It is consistent that MRP holds and NS_{ω_1} is saturated (for example, under MM).
- In the model, no ccc forcing adds a TCG-sequence.
- By adding a bottom to a Q -point, we can force KA without adding a TCG-sequence.

$(+)_{<\omega}$ and $(+)^p$

Theorem (I) $\text{Add}(\omega_1, \aleph_2)$ forces $(+)_{<\omega}$ and $\neg(+)^p(S)$ for every stationary subset S of ω_1 .

Proof(Sketch)

- Let $P = \text{Add}(\omega_1, \aleph_2)$. $(+)_{<\omega}$ is easy. Suppose $(+)^p(S)$ is forced.
- For every $p \in P$, there are typical countable elementary submodels M and N with $P, p, S, \dots \in N \cap M$, $M \cap \omega_1 = N \cap \omega_1 \in S$, $(M \setminus N) \cap \omega_2 \neq \emptyset$, and $(N \setminus M) \cap \omega_2 \neq \emptyset$.
- So, M and N have totally independent copies of $\text{Add}(\omega_1, 1)$.
- By using these independent copies, we can build a $q \leq p$, $\dot{D} \in M$, and $\dot{E} \in N$ such that q is (M, P) -generic and (N, P) -generic, and q forces that \dot{D} and \dot{E} are club subsets of ω_1 , but $\dot{D} \cap \dot{E} \cap M = \emptyset$.
Contradiction.

Strong club guessing sequences

Definition Let S be a stationary subset of an uncountable regular cardinal κ . A sequence $\langle C_\delta : \delta \in S \rangle$ is a *strong club guessing (SCG) sequence* iff

- (i) each C_δ is an unbounded subset of δ , and
- (ii) for every club subset D of κ , there exist a club subset of $\delta \in S$ such that $C_\delta \subseteq^* D$.

Facts

- The existence of an SCG-sequence on S implies $(+)^p_{<\omega}(S)$.
- $V = L$ implies that for every uncountable regular cardinal κ , κ is not ineffable iff κ carries an SCG-sequence.

$$\diamond^+ \Rightarrow \exists \text{SCG?}$$

Definition A sequence $\langle A_\delta : \delta < \omega_1 \rangle$ is a \diamond^+ -sequence iff each A_δ is a countable subset of $\mathcal{P}(\delta)$ and for every subset X of ω_1 , there exists a club subset D of ω_1 such that for every $\delta \in D$, $X \cap \delta, D \cap \delta \in A_\delta$.

Question Does \diamond^+ imply the existence of an SCG-sequence?

Theorem (I) $\diamond^+ + \neg(+)^p_{<\omega}$ is consistent. In particular, it is consistent that \diamond^+ holds but there is no SCG-sequence on ω_1 .

Proof(Sketch)

- Just use the standard forcing to add a \diamond^+ -sequence. Namely, add a sequence by countably closed forcing and keep shooting clubs to make it a \diamond^+ -sequence.
- As before, we can show that it forces $\neg(+)^p_{<\omega}$.

Open Questions

Question Is $(+)_{<\omega}$ equivalent to $(+)$?

Question Does $\neg(+)<\omega$ imply that no ccc forcing can add a TCG-sequence?

Question Does CH imply $(+)$?

Question Does $(+)$ or $(+)_{<\omega}$ imply the club guessing principle?
(Conjecture: No)

Question What is the consistency strength of $\neg(+)$?

Question Does \diamond^+ imply the existence of an SCG-sequence on some stationary subset of ω_1 ? Does it imply $(+)^p(S)$ for some stationary subset of ω_1 ?