

# The Comparison of Various Club Guessing Principles.

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## Club subsets of $\omega_1$

$\omega_1$  denotes the least uncountable ordinal. We adopt Von Neumann's definition, so it is the set of all countable ordinals.

A subset  $D$  of  $\omega_1$  is closed unbounded (club) if and only if

- (i) for every  $\gamma < \omega_1$ , there exists a  $\delta \in D$  such that  $\gamma < \delta$ , and
- (ii) for every  $\gamma < \omega_1$ , if  $D \cap \gamma$  is unbounded in  $\gamma$ , then  $\gamma \in D$ .

For example, assume the Continuum Hypothesis (CH). Then, there is an enumeration  $\langle r_\alpha : \alpha < \omega_1 \rangle$  of  $\mathbb{C}$ .

Let  $D$  be the set of all  $\delta < \omega_1$  such that  $\{r_\alpha : \alpha < \delta\}$  is an algebraically closed subfield of  $\mathbb{C}$ . Then,  $D$  is a club subset of  $\omega_1$ .

**Proposition** The intersection of countably many club subsets of  $\omega_1$  is also club.

## Club guessing sequences

Let  $\text{Lim}$  stand for the class of limit ordinals.  $X \subseteq^* Y$  means that there exists a  $\zeta < \sup(X)$  such that  $X \setminus \zeta \subseteq Y$ .

**Definition**(S. Shelah) A sequence  $\vec{C} = \langle C_\gamma : \gamma \in \omega_1 \cap \text{Lim} \rangle$  is called a *fully club guessing (FCG) sequence on  $\omega_1$*  iff

- (i) each  $C_\gamma$  is an unbounded subset of  $\gamma$ , and
- (ii) for every club subset  $D$  of  $\omega_1$ , there exists a  $\delta \in \omega_1 \cap \text{Lim}$  such that  $C_\delta \subseteq D$ .

For example,  $V = L$  implies the existence of an FCG-sequence.

Let CG denote “there exists an FCG-sequence on  $\omega_1$ ”.

## Kunen's Axiom

Recall that  $\langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  is a *ladder system* if and only if each  $C_\delta$  is an unbounded subset of  $\delta$  of order type  $\omega$ .

**Definition**(K. Kunen) *Kunen's Axiom (KA)* asserts that there exists a ladder system  $\langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  such that for every club subset  $D$  of  $\omega_1$ , there exists an  $\delta \in \omega_1 \cap \text{Lim}$  such that for all but finitely many  $n < \omega$ ,  $D \cap [C_\delta(n), C_\delta(n+1)) \neq \emptyset$ . Here,  $C_\delta(n)$  denotes the  $(n+1)$ -st element of  $C_\delta$ .

### Fact

- If there exists a FCG-sequence on  $\omega_1$ , then KA holds.
- We may replace  $D \cap [C_\delta(n), C_\delta(n+1))$  by  $D \cap (C_\delta(n), C_\delta(n+1))$ .

## The Principle $\mathcal{U}$

**Definition**(S. Todorcevic) Let  $k \leq \omega$ .  $\mathcal{U}_k$  is defined to be the principle that asserts the existence of a sequence  $\langle f_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  such that

- for each  $\delta \in \omega_1 \cap \text{Lim}$ ,  $f_\delta$  is a continuous function from  $\delta$  into  $k$ , and
- for every club subset  $D$  of  $\omega_1$ , there exists a  $\delta \in \omega_1 \cap \text{Lim}$  such that for every  $\zeta < \delta$ ,  $f_\delta''(D \cap [\zeta, \delta)) = k$ .

Such a sequence is called a  $\mathcal{U}_k$ -sequence.  $\mathcal{U}$  means  $\mathcal{U}_\omega$ .

### Fact

- KA implies  $\mathcal{U}$
- For every  $k < l \leq \omega$ ,  $\mathcal{U}_l$  implies  $\mathcal{U}_k$ .
- In case of  $\mathcal{U}$ , we can replace  $f_\delta''(D \cap [\zeta, \delta)) = \omega$  by  $f_\delta''(D \cap \delta) = \omega$ .

## The Small Jump Axiom

**Definition**(P. Nyikos) The *Small Jump Axiom* (SJA) is the principle that asserts the existence of a ladder system  $\langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  such that for every club subset  $D$  of  $\omega_1$ , there exist successive members  $\xi$  and  $\xi'$  of  $D$  and  $\delta \in \omega_1 \cap \text{Lim}$  such that  $|\llbracket \xi, \xi' \rrbracket \cap C_\delta| = 1$ .

The name ‘The Small Jump Axiom’ is due to me, but I will really appreciate if somebody can give it a better name (or tell me it was already done).

**Fact** KA implies SJA.

## $\mathcal{U}_2$ and SJA

**Theorem (P. Nyikos)**  $\mathcal{U}_2$  is equivalent to the following principle: there exists a ladder system  $\langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  such that for every club subset  $D$  of  $\omega_1$ , there exist a  $\delta \in \omega_1 \cap \text{Lim}$  and successive members  $\xi$  and  $\xi'$  of  $D$  such that  $|\llbracket \xi, \xi' \rrbracket \cap D|$  is odd.

In fact, this is also equivalent to a topological statement.

**Corollary** SJA implies  $\mathcal{U}_2$ .

## The Principle (+)

**Definition (P. Larson)** (+) is the principle that asserts the existence of a stationary subset  $S$  of  $[H(\aleph_2)]^{\aleph_0}$  such that for every  $M, N \in S$  with  $M \cap \omega_1 = N \cap \omega_1$ , for every pair of club subsets  $D \in M$  and  $E \in N$  of  $\omega_1$ , we have  $D \cap E \cap M \neq \emptyset$ .

$(+)_{<\omega}$  is the version of (+) that allows any finite number of models. Namely,

**Definition (P. Larson)**  $(+)_{<\omega}$  is the principle that asserts the existence of a stationary subset  $S$  of  $[H(\aleph_2)]^{\aleph_0}$  such that for every  $N_1, \dots, N_n \in S$ , if  $N_i \cap \omega_1 = N_0 \cap \omega_1$  for every  $i$ , whenever  $D_i \in N_i$  is a club subset of  $\omega_1$  for each  $i$ , we have  $\bigcap_i D_i \cap M \neq \emptyset$ .

## Alternative Definition

**Lemma**  $(+)$  is equivalent to the existence of a sequence  $\langle F_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  (called a  $(+)$ -sequence) such that

- $F_\delta$  is a upward closed family of unbounded subsets of  $\delta$ ,
- for every  $x, y \in F_\delta$ ,  $x \cap y$  is unbounded in  $\delta$ , and
- for every club subset  $D$  of  $\omega_1$ , there exists a  $\delta \in \omega_1 \cap \text{Lim}$  such that  $D \cap \delta \in F_\delta$ .

**Lemma (P. Larson)**  $(+)_{<\omega}$  is equivalent to the existence of a sequence  $\langle F_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  (called a  $(+)_{<\omega}$ -sequence) such that

- $F_\delta$  is a filter on  $\delta$ , and
- for every club subset  $D$  of  $\omega_1$ , there exists a  $\delta \in \omega_1$  such that  $D \cap \delta \in F_\delta$ .

## Weak Club Guessing Sequences

**Definition (S. Shelah)** A sequence  $\vec{C} = \langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  is a *weak club guessing sequence* (wCG-sequence) if and only if

- each  $C_\delta$  is an unbounded subset of  $\delta$ , and
- for every club subset  $D$  of  $\omega_1$ , there exists a  $\delta \in \omega_1 \cap \text{Lim}$  such that  $D \cap C_\delta$  is unbounded in  $\delta$ .

$\vec{C}$  is called *short* if and only if for almost every  $\delta \in \omega_1 \cap \text{Lim}$ ,  $\text{otp}(C_\delta) < \delta$ .

$\vec{C}$  has *order type*  $\alpha$  if and only if for almost every  $\delta \in \omega_1 \cap \text{Lim}$ ,  $\text{otp}(C_\delta) = \alpha$ .

## $(+)_{<\omega}$ and Weak Club Guessing

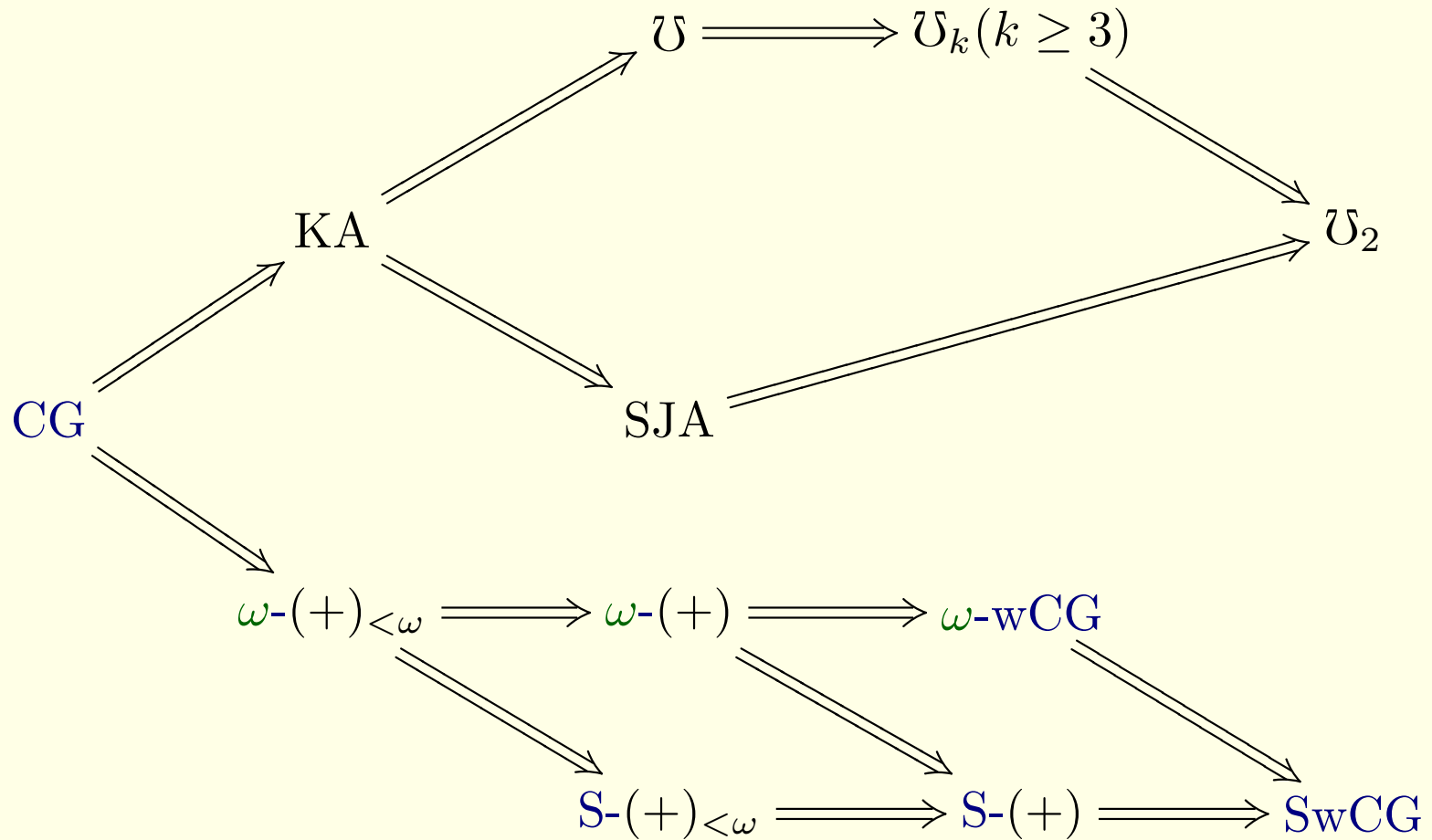
### Definition

- A  $(+)$ -sequence  $\langle F_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  is *short* if and only if for almost every  $\delta \in \omega_1 \cap \text{Lim}$ , there exists an  $x \in F_\delta$  such that  $\text{otp}(x) < \delta$ .
- A  $(+)$ -sequence  $\langle F_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  has order type  $\alpha$  if and only if for almost every  $\delta \in \omega_1 \cap \text{Lim}$ , there exists an  $x \in F_\delta$  such that  $\text{otp}(x) = \alpha$ .

### Proposition

- If there is a short  $(+)$ -sequence, then there is a short weak club guessing sequence.
- If there is a  $(+)$ -sequence of order type  $\alpha$ , then there is a weak club guessing sequence of order type  $\alpha$ .

# Diagram



“ $\omega$ -” means order type  $\omega$  and “S-” means “short”.

## Question

Question Can we add any more arrows?

Answer: Almost no. Except between (+) variations, there are no other implications.

The rest of this talk is devoted to giving outlines of the proofs.

## Totally proper forcing

**Definition** A forcing notion  $P$  is *totally proper* if and only if it is proper and adds no new reals.

$p \in P$  is *totally*  $(N, P)$ -*generic* if and only if  $p$  is  $(N, P)$ -generic and decides every dense subset of  $P$  lying in  $N$ .

A forcing notion  $P$  is  $\omega^\omega$ -*bounding* if and only if for every function  $f : \omega \rightarrow \omega$  in the extension by  $P$ , there exists a  $g : \omega \rightarrow \omega$  lying in the ground model such that  $f \leq g$ .

$p$  is *finitely*  $(N, P)$ -*generic* if and only if  $p$  is  $(N, P)$ -generic and for every  $P$ -name  $\dot{\tau} \in N$  for ordinals, there exists a finite set  $\{\gamma_0, \dots, \gamma_{k-1}\}$  of ordinals such that  $p \Vdash \dot{\tau} = \gamma_i$  for some  $i < k$ .

## Some Iteration Lemmas

Lemma (S. Shelah)  $\omega^\omega$ -bounding proper forcing is preserved under countable support iteration.

Lemma (S. Shelah) If a countable support iteration

$\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \leq \eta \rangle$  satisfies that for every  $\alpha < \eta$ ,  $P_\alpha$  forces

- $\dot{Q}_\alpha$  is completely proper (stronger than just being totally proper), and
- either  $\dot{Q}_\alpha$  is weakly  $<\omega_1$ -proper or it is proper in every totally proper extension,

then  $P_\eta$  is totally proper.

Total properness itself is not preserved under countable support iteration. The requirement of complete properness is justified by the weak diamond. It is also shown by S. Shelah that complete properness is not sufficient.

## Adding one Cohen Real

All forcing notions we use in this talk satisfy the conditions of Shelah's theorem.

**Theorem** It is consistent that none of the above principles holds (with CH or  $2^{\aleph_0} = \aleph_2$ ).

**Proof** Just kill all  $\mathcal{U}_2$ -sequences and short wCG in the standard way.

**Lemma** After adding one Cohen real,

- $\mathcal{U}$ , SJA, and  $\omega$ -wCG hold.
- KA fails if it fails in the ground model.
- (+) fails if it fails in the ground model.

So, we can obtain a model of  $\text{CH} + \neg \text{KA} + \neg(+)$  +  $\mathcal{U}$  + SJA +  $\omega$ -wCG.

## Generalized Club Guessing Sequence

**Definition** A sequence  $\vec{I} = \langle I_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  is called a *generalized club guessing sequence* if and only if

- for every  $\delta \in S$ ,  $I_\delta$  is a downward closed family of subsets of  $\delta$  such that for every  $\zeta < \delta$  and  $x \in I_\delta$ ,  $x \cup \zeta \in I_\delta$ .
- for every club subset  $D$  of  $\omega_1$ , there exists a  $\delta \in S$  such that  $D \cap \delta \notin I_\delta$

Note that we do not require any intersection property to  $I_\delta$  although typically  $I_\delta$  forms an ideal.

We say that  $\vec{I}$  is *countably generated* if and only if there exists a countable set  $X \subseteq I_\delta$  such that for every  $y \in I_\delta$ , there exists an  $x \in X$  such that  $y \subseteq x$ .

## Some Examples of GCG

**Example** Let  $\vec{f} = \langle f_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  be a sequence such that  $f_\delta : \delta \rightarrow \omega$  is continuous for every  $\delta \in \omega_1 \cap \text{Lim}$ . For each  $\delta \in \omega_1 \cap \text{Lim}$ , define  $I_\delta \subseteq \mathcal{P}(\delta)$  be defined by  $x \in I_\delta$  if and only if there exists a  $\zeta < \delta$  such that  $f''_\delta(x \setminus \zeta) \neq \omega$ . Let  $\vec{I} = \langle I_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$ . Then,  $\vec{f}$  is a  $\mathcal{U}$ -sequence if and only if  $\vec{I}$  is a generalized club guessing sequence. Moreover,  $\vec{I}$  is countably generated.

**Example** Let  $\vec{C} = \langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  be a guessing sequence. Define  $\vec{I} = \langle I_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  as follows: for every subset  $x$  of  $\delta$ ,  $x \in I_\delta$  if and only if  $C_\delta \cap x$  is bounded in  $\delta$ . Then,  $\vec{C}$  is a weak club guessing sequence if and only if  $\vec{I}$  is a generalized club guessing sequence.  $\vec{I}$  is countably generated.

## Preservation of Countably Generated GCG

For the moment, let  $\vec{I} = \langle I_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  be a countably generated generalized club guessing sequence and  $\theta$  a sufficiently large regular cardinal.

**Definition** We say that a countable elementary submodel  $N$  of  $H(\theta)$  is  $\vec{I}$ -good if and only if for every club subset  $D$  of  $\omega_1$ ,  $D \cap N \notin I_{N \cap \omega_1}$ .

**Lemma** There exists stationarily many  $M$  in  $[H(\theta)]^{\aleph_0}$  that is  $\vec{I}$ -good.

**Definition** We say that a forcing notion  $P$  is  $\vec{I}$ -proper if and only if for every  $\vec{I}$ -good countable elementary submodel  $N$  of  $H(\theta)$  with  $P \in N$  for every  $p \in P \cap N$ , there exists a  $q \leq p$  that is  $(N, P)$ -generic and  $q \Vdash 'N[\dot{G}] \text{ is } \vec{I}\text{-good}'$ .

**Lemma**  $\vec{I}$ -properness is preserved under countable support iteration.

# Applications

**Theorem** Each of the following is consistent.

- $\text{CH} + \mathcal{U} + \neg \text{SJA}$
- For every  $k < \omega$ ,  $\text{CH} + \mathcal{U}_k + \neg \mathcal{U}_{k+1}$
- $\text{CH} + \text{SwCG} + \neg \omega\text{-wCG}$ .

The witnessing models can be built by the standard iterated forcing argument with the iteration theorem about generalized club guessing sequences.

What about the principles which cannot be described in terms of generalized club guessing sequences?

## $d$ -dimensional Generalized Club Guessing Sequences

**Definition** Let  $d \in [1, \omega)$ . A sequence  $\vec{I} = \langle I_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  is called a  $d$ -dimensional generalized club guessing sequence if and only if

- for every  $\delta \in \omega_1 \cap \text{Lim}$ ,  $I_\delta$  is a downward closed family of subsets of  $[\delta]^d$  such that for every  $\zeta < \delta$  and  $x \in I_\delta$ ,  $x \cup [\zeta]^d \in I_\delta$ .
- for every club subset  $D$  of  $\omega_1$ , there exists a  $\delta \in \omega_1 \cap \text{Lim}$  such that  $[D \cap \delta]^d \notin I_\delta$

We say that  $\vec{I}$  is *countably generated* if and only if there exists a countable set  $X \subseteq I_\delta$  such that for every  $y \in I_\delta$ , there exists an  $x \in X$  such that  $y \subseteq x$ .

## SJA as 2-dimensional GCG

**Example** Let  $\vec{C} = \langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  be a ladder system. For each  $\delta \in \omega_1 \cap \text{Lim}$ , let  $I_\delta$  be the set of all subsets  $x$  of  $[\delta]^2$  such that there exists a  $\zeta < \delta$  such that for every  $\{\xi, \xi'\} \in x$  with  $\zeta \leq \xi < \xi'$ ,  $|C_\delta \cap [\xi, \xi')| \neq 1$ . Define  $\vec{I} = \langle I_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$ . Then,  $\vec{C}$  is an SJA-sequence if and only if  $\vec{I}$  is a 2-dimensional generalized club guessing sequence. Moreover,  $\vec{I}$  is countably generated.

By using this definition, we can use the same argument as in the case of (1-dimensional) generalized club guessing sequences to show the following theorem.

**Theorem**  $\text{CH} + \text{SJA} + \neg \mathcal{U}_3$  is consistent.

What if the principle cannot be described as a countably generated  $d$ -dimensional generalized club guessing sequence?

## Preservation of $\omega^\omega$ -bounding

The following lemma is the key for the preservation of  $\omega^\omega$ -bounding.

**Lemma**(S. Shelah) Suppose that

- $P$  is an  $\omega^\omega$ -bounding proper forcing notion,
- $\dot{Q}$  is a  $P$ -name for an  $\omega^\omega$ -bounding proper forcing notion,
- $N_0$  and  $N_1$  are both countable elementary submodels of  $H(\theta)$  with  $P, \dot{Q} \in N_0 \in N_1$ ,
- $p$  is finitely  $(N_0, P)$ -generic, and  $(N_1, P)$ -generic, and
- $\dot{q} \in N_0$  is a  $P$ -name for an element of  $\dot{Q}$ .

Then, there exists a  $P$ -name  $\dot{q}'$  such that  $p \Vdash \dot{q}' \leq \dot{q}$  and  $(p, \dot{q}')$  is finitely  $(N_0, P * \dot{Q})$ -generic and  $(N_1, P * \dot{Q})$ -generic.

That is, if  $p \in P$  is ‘strong enough’, then there is a pure extension of  $p$  that is strong enough in  $P * \dot{Q}$ .

## $(+)_{<\omega}$ -proper forcing

**Definition** Let  $P$  be a forcing notion and  $\vec{F} = \langle F_\xi : \xi \in \omega_1 \cap \text{Lim} \rangle$  a  $(+)_{<\omega}$ -sequence. We say that  $P$  is  $(+)_{<\omega}$ -proper for  $\vec{F}$  if and only if whenever

- $\langle N_\gamma : \gamma \leq \delta + 1 \rangle$  is a tower of countable subsets of  $H(\theta)$  such that  $N_\delta \cap \omega_1 = \delta$  and  $N_\delta$  and  $N_{\delta+1}$  are elementary submodels of  $H(\theta)$ ,  $\vec{F}, P \in N_0$ ,
- $p \in P \cap N_0$ ,
- $x \in F_\delta$  with  $\text{ot}(x) = \omega$ ,
- for every  $\gamma \in x$ ,  $N_\gamma \cap \omega_1 = \gamma$  and  $N_\gamma$  is an elementary submodel of  $H(\theta)$ ,
- for every  $y \in F_\delta \cap N_{\delta+1}$ ,  $x \subseteq^* y$ ,

there exists a  $q \leq p$  such that  $q$  is  $(N_\gamma, P)$ -generic for every  $\gamma \in X$ .

## Key Lemma for $(+)_{<\omega}$

**Lemma** Suppose

- $P$  is  $(+)_{<\omega}$ -proper and  $\dot{Q}$  is forced to be  $(+)_{<\omega}$ -proper.
- $\langle N_\gamma : \gamma \leq \delta + 1 \rangle$  is a tower of countable subsets of  $H(\theta)$  such that  $N_\delta \cap \omega_1 = \delta$ ,  $N_\delta$ ,  $N_{\delta+1}$ , and  $N_{\delta+2}$  are elementary submodels of  $H(\theta)$ ,
- $\dot{q} \in N_0$  is a  $P$ -name for an element of  $\dot{Q}$ ,
- $x \in F_\delta$  with  $\text{ot}(x) = \omega$ ,
- for every  $\gamma \in x$ ,  $N_\gamma \cap \omega_1 = \gamma$  and  $N_\gamma$  is an elementary submodel of  $H(\theta)$ ,
- for every  $y \in F_\delta \cap N_{\delta+1}$ ,  $x \subseteq^* y$ ,
- $p \in P$  is finitely  $(N_\gamma, P)$ -generic for every  $\gamma \in x$ , finitely  $(N_\delta, P)$ -generic, finitely  $(N_{\delta+1}, P)$ -generic, and  $(N_{\delta+2}, P)$ -generic.

Then, there exists a  $P$ -name  $\dot{q}'$  of an element of  $\dot{Q}$  such that  $p \Vdash \dot{q}' \leq \dot{q}$  and  $(p, \dot{q}')$  is finitely  $(N_\gamma, P * \dot{Q})$ -generic for every  $\gamma \in x$ , finitely  $(N_\delta, P * \dot{Q})$ -generic, finitely  $(N_{\delta+1}, P * \dot{Q})$ -generic, and  $(N_{\delta+2}, P * \dot{Q})$ -generic.

## Preservation of $(+)_{<\omega}$ -proper forcing

**Lemma** If  $\vec{F}$  is a ‘good’  $(+)_{<\omega}$ -sequence of order type  $\omega$ , then  $(+)_{<\omega}$ -properness for  $\vec{F}$  is preserved under countable support iteration.

**Theorem**  $\text{CH} + (+)_{<\omega} + \neg\mathcal{U}_2$  is consistent. So, there is no implication from the lower half to the upper half.

**Question** Does the  $(+)_{<\omega}$ -sequence need to be ‘good’ and of order type  $\omega$ ?

**Question** Is  $\text{CH} + \text{S-}(+)_{<\omega} + \neg\omega\text{-}(+)_{<\omega}$  consistent?

**Question** Is there a preservation theorem for  $(+)$ ?

## Preservation of KA

Let  $\langle \theta_n : n < \omega \rangle$  be an increasing sequence of large regular cardinals and  $\theta$  a regular cardinal with  $\theta > \sup_{n < \omega} \theta_n$ .

Temporarily, we say that  $P$  satisfies  $(*)$  if and only if for every typical  $N$ , there exists a  $f : \omega \rightarrow \omega$  such that whenever

- $g : \omega \rightarrow \omega$  and  $g \geq f$ ,
- $\langle M_n : \bar{n} \leq n < \omega \rangle$  is a tower of countable sets with  $M_n \prec H(\theta_{g(n)})$ ,  
 $M_n \cap \omega_1 \in [C_\delta(n), C_\delta(n+1))$  where  $\delta = \bigcup_{\bar{n} \leq n < \omega} M_n$ ,
- $N \cap H(\theta_0) = \bigcup_{\bar{n} \leq n < \omega} M_n \cap H(\theta_0)$ ,
- $p \in P \cap M_{\bar{n}}$ , and
- some more minor assumptions,

there exist a  $q \leq p$  and a tower  $\langle M'_n : \bar{n} \leq n < \omega \rangle$  such that  $M'_n \prec H(\theta_{g(n)-f(n)})$  and  $q$  is  $(M'_n, P)$ -generic for every  $n \in [\bar{n}, \omega)$ .

## Application

Theorem  $\text{CH} + \text{KA} + \neg \text{SwCG}$  is consistent. So, there is no implication from the upper half to the lower half.

Theorem The diagram is complete except (+)-variations.

## Open Questions

Question Are there any non-trivial relations between (+) variations?

Question Is the dominating number  $\mathfrak{d}$  related to these principles?

Question Are there any non-trivial club guessing principles that follow from CH?

Question Does measuring imply CH? Measuring is the principle that asserts whenever  $\langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$  is a sequence such that each  $C_\delta$  is a club subset of  $\delta$ , there exists a club subset  $D$  of  $\omega_1$  such that for every limit point  $\delta$  of  $D$ , there exists a  $\zeta < \delta$  such that either  $D \cap [\zeta, \delta) \subseteq C_\delta$  or  $D \cap [\zeta, \delta) \cap C_\delta = \emptyset$ .