NOTES ON SUB-OSTASZEWSKI SPACES

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0. Introduction

This is a rough sketch on the difficulty to obtain a model of CH without any locally compact, sub-Ostaszewski spaces. In [1], Eisworth and Roitman defined a forcing notion to kill a given locally compact sub-Ostaszewski space (ω_1, τ) without adding reals. It does so by adding an unbounded τ-closed subset of ω_1 whose all initial segment have τ-compact closure. They proved the consistency with CH that there is no Ostaszewski space by showing the forcing notion defined from an Ostaszewski space can be iterated without adding reals.

However, it is still open whether it is consistent with CH that there is no locally compact, sub-Ostaszewski space. In this note, we shall show that, for example, if we begin with L and use forcing to obtain the witnessing model, then there exists a locally compact, sub-Ostaszeski space in the intermediate model which is killed in a different way.

1. Definition

$\mathrm{lp}(\alpha)$ denotes the limit part of α. Lim stands for the class of all limit ordinals and Lim^2 stands for the class of all limit of limit ordinals, i.e. ordinals divisible by ω^2. For a set X of ordinals, $\lim(X)$ is defined as the set of all limit points of X.

The following definition gives a clear formulation of the standard construction of a locally compact, thin tall space. It is needed to define a family of sub-Ostaszewski space in Section 2.

Let $\mathcal{F}$ be the set of all functions f such that $\mathrm{dom}(f) \leq \omega_1$, and for every $\alpha \in \mathrm{dom}(f)$,

(i) $\alpha \in f(\alpha)$,
(ii) $f(\alpha) \setminus \{\alpha\} \subseteq \mathrm{lp}(\alpha)$,
(iii) for every $n < \omega$, $f(\alpha) \cap f(\alpha + n) = \varnothing$,
(iv) for every $\beta < \alpha$, if $\beta \in f(\alpha)$, then there exists a finite subset s of β such that $f(\beta) \setminus \bigcup_{\gamma \in s} f(\gamma) \subseteq f(\alpha)$, and
(v) for every $\beta < \alpha$, if $\beta \notin f(\alpha)$, then there exists a finite subset s of β such that $f(\beta) \setminus \bigcup_{\gamma \in s} f(\gamma)$ is disjoint from $f(\alpha)$.

Date: September 30, 2005.

Let $f \in \mathcal{F}$. Let $\tau(f)$ be a topology on $\omega \operatorname{dom}(f)$ whose basic open sets are of the form $f(\alpha) \setminus \bigcup_{\beta \in z} f(\beta)$ where $\alpha \in \operatorname{dom}(f)$ and z is a finite subset of α.

Lemma 1.1. *Let $f \in \mathcal{F}$. Then $(\operatorname{dom}(f), \tau(f))$ is a regular, Hausdorff, locally compact space.*

Proof. Let $\tau = \tau(f)$. It is clearly Hausdorff. Notice that if $\alpha \in \operatorname{dom}(f)$ and z is a finite subset of α, then $f(\alpha) \setminus \bigcup_{\beta \in z} f(\beta)$ is τ-clopen. The regularity easily follows from this fact. We shall show that for every $\alpha \in \operatorname{dom}(f)$, $f(\alpha)$ is τ-compact by induction. Suppose that for every $\beta < \alpha$, $f(\beta)$ is τ-compact. Let $\mathcal{C}$ be a τ-open cover of $f(\alpha)$. We may assume that all elements of $\mathcal{C}$ are basic open sets. Then there exists a $U \in \mathcal{C}$ such that $\alpha \in U$. Hence, there exists a finite subset z of α such that $U = f(\alpha) \setminus \bigcup_{\beta \in z} f(\beta)$. But for each $\beta \in z$, since $f(\beta)$ is τ-compact, there exists a finite subset $\mathcal{C}_\beta$ of $\mathcal{C}$ which covers $f(\beta)$. Therefore, $U \cup \bigcup_{\beta \in z} \mathcal{C}_\beta$ is a finite subcover of $\mathcal{C}$. □(Lemma1.1)

2. Coding by sub-Ostaszewski spaces

Let T be the set of all functions t from some $\alpha \in (0, \omega_1)$ into 2 such that $t(0) = 0$.

Lemma 2.1. *Suppose that $V = L[A]$ for some $A \subseteq \omega_1$. Then there exist functions φ and ψ satisfying the following conditions.*

- (i) (a) $\operatorname{dom}(\varphi) = T$,
  - (b) *for every $t \in T$, $\varphi(t)$ is a function with domain $\omega \operatorname{dom}(t)$ which defines a thin topology, and*
  - (c) *if $t_1 \subseteq t_2$ are both in T, then $\varphi(t_1) \subseteq \varphi(t_2)$.*
- (ii) (a) $\operatorname{dom}(\psi) = T \times 2$, *and*
  - (b) *for every $t \in T$, $\psi(t, 0)$ and $\psi(t, 1)$ are pairwise disjoint $\tau(\varphi(t))$-clopen sets such that $\psi(t, 0) \cup \psi(t, 1) = \omega \operatorname{dom}(t)$.*
- (iii) *$\varphi(t)$, $\psi(t, 0)$, and $\psi(t, 1)$ are uniformly definable from t and $A \cap \omega \operatorname{dom}(t)$ in $L[A \cap \omega \operatorname{dom}(t)]$.*
- (iv) *For every cofinal branch f of T, let $\varphi(f) = \bigcup\{\varphi(f \restriction \alpha) : 0 < \alpha < \omega_1\}$. Then,*
  - (a) *$\tau(\varphi(f))$ is a sub-Ostaszewski topology, and*
  - (b) *for every $\alpha \in (0, \omega_1)$, if x is an unbounded subset of $\omega\alpha$ of order type ω and has a $\tau(\varphi(f))$-compact closure, then $x \subseteq^* \psi(f \restriction \alpha, f(\alpha))$.*

Proof. Let $\vec{C} = \langle C_\delta : \delta \in \omega_1 \cap \operatorname{Lim} \rangle$ be a ♣-sequence such that

- each C_δ has order type ω (let $\langle c_{\delta,n} : n < \omega \rangle$ be the increasing enumeration of C_δ)
- If $\delta = \bar{\delta} + \omega$ for some limit $\bar{\delta} < \delta$, then $C_\delta \cap \bar{\delta} = \varnothing$,
- If $\delta \in \operatorname{Lim}^2$, then $|c_{\delta,n} \cap \operatorname{Lim}| > n$, and
- C_δ is uniformly definable from δ and $A \cap \delta$ in $L[A \cap \delta]$.

For $\delta \in \omega_1 \cap \mathrm{Lim}$ and $i = 0, 1$, define $C^i_\delta = \{c_{\delta,2n+i} : n < \omega\}$. Then for every cofinal branch f in T, $\langle C^{f(\delta)}_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ is a definable ♣-sequence.

We shall inductively define φ and ψ. We shall also define $U(t, n)$ for $t \in T$ and $n < \omega$ such that for every $t \in T$, if we let $\alpha = \mathrm{dom}(t)$,

(v) $\{U(t, n) : n < \omega\}$ is a pairwise disjoint family of $\tau(\varphi(t))$-compact open sets which are bounded in $\omega\alpha$,
(vi) for every $n, m < \omega$, $c_{\omega\alpha,m} \in U(t, n)$ if and only if $m = n$,
(vii) $\bigcup_{n<\omega} U(t, n) = \omega\alpha$,
(viii) $\langle U(t, n) : n < \omega\rangle$ is uniformly definable from t and $A \cap \omega\alpha$ in $L[A \cap \omega\alpha]$,
(ix) for every $\beta \in (0, \alpha)$ and $n < \omega$, there are infinitely many $l < \omega$ such that $U(t \restriction \beta, 2l + t(\beta)) \cap \varphi(t)(\omega\beta + n) \neq \varnothing$, and
(x) for every $\beta \in (0, \alpha)$ and $\xi \in [\omega\beta, \omega\alpha)$, there are at most finitely many $l < \omega$ such that $U(t \restriction \beta, 2l + 1 - t(\beta)) \cap \varphi(t)(\xi) \neq \varnothing$.
(xi) for every $\beta \in (0, \alpha)$ and $n < \omega$, there are at most finitely many $l < \omega$ such that $U(t \restriction \beta, 2l + 1 - t(\beta)) \cap U(t, n) \neq \varnothing$.

When $U(t, n)$ has been defined for all $n < \omega$, let $\psi(t, i) = \bigcup\{U(t, 2l + i) : l < \omega\}$.

Let $\langle a_n : n < \omega\rangle$ be the $<_J$-least partition of ω into $\aleph_0$-many infinite pieces such that $a_n \cap n = \varnothing$.

First let $t = \{\langle 0, 0\rangle\}$. Define for every $n < \omega$, $\varphi(t)(n) = \{n\}$ and $U(t, n) = \{c_{\omega,n}\}$.

Let $t \in T$ and $\alpha = \mathrm{dom}(t)$ be such that $\alpha > 1$. First suppose that we have defined $\varphi(t \restriction \beta)$, $\psi(t \restriction \beta, i)$, and $U(t \restriction \beta, n)$ for every $\beta \in (0, \alpha)$, $i = 0, 1$, and $n < \omega$. We shall define $\varphi(t)$.

Case 1. $\alpha = \bar\alpha + 1$ for some $\bar\alpha$.

Let $\langle \beta_k : k < \omega\rangle$ be the $<_J$-least enumeration of $\bar\alpha$ such that $\omega\bar\alpha_k < c_{\omega\bar\alpha,k}$. We allow redundancy although it does not matter unless $\bar\alpha < \omega$. Let $\bar t = t \restriction \bar\alpha$.

Claim 1. For every $m, n < \omega$, $U(\bar t, m) \setminus \bigcup_{k\le n} \bigcup_{l<\omega} U(t \restriction \beta_k, 2l + 1 - t(\beta_k))$ is $\tau(\varphi(\bar t))$-compact open.

$\vdash$ Let $k \le n$. Since $k \le m$, we have $\gamma_k < c_{\omega\bar\beta,k} \le c_{\omega\bar\beta,m}$. By inductive hypothesis, there are only finitely many $l < \omega$ such that $U(\bar t \restriction \beta_k, 2l + 1 - t(\beta_k)) \cap U(\bar t, m) \neq \varnothing$. Therefore, $U(\bar t, m) \setminus \bigcup_{l<\omega} U(\bar t \restriction \beta_k, 2l + 1 - t(\beta_k))$ is a $\tau(\varphi(\bar t))$-compact open set.

Since it is a finite intersection of $\tau(\varphi(\bar t))$-compact open sets, $U(\bar t, m) \setminus \bigcup_{k\le n} \bigcup_{l<\omega} U(\bar t \restriction \beta_k, 2l + 1 - t(\beta_k))$ is $\tau(\varphi(\bar t))$-compact open. $\dashv$ (Claim1)

For each $n < \omega$, define

$$\varphi(t)(\omega\bar{\alpha}+n) = \{\omega\bar{\alpha}+n\}\cup \bigcup_{m\in a_n}\left(U(\bar{t}, 2m+t(\bar{\delta}) \setminus \bigcup_{k\leq n}\bigcup_{l<\omega} U(t\restriction\beta_k, 2l+1-t(\beta_k)))\right)$$

It is easy to see that it satisfies the inductive hypothesis.

Case 2. $\alpha \in \mathrm{Lim}^2$.

In this case, let $\varphi(t) = \bigcup\{\varphi(t\restriction\beta) : \beta < \alpha\}$.

Suppose that $\varphi(t)$, $\psi(t\restriction\beta, i)$, and $U(t\restriction\beta, n)$ have been defined for $\beta < \alpha$. We shall define $U(t,n)$ by induction on n. Let $\langle\beta_n : n < \omega\rangle$ be the $<_J$-least enumeration of α and $\langle\xi_n : n < \omega\rangle$ the $<_J$-least enumeration of $\omega\alpha$. Suppose that $U(t,m)$ has been defined for every $m < n$. If $\xi_n \in C_{\omega\alpha}\cup\bigcup_{m<n} U(t,m)$, then let

$$W = \bigcup\{U(t\restriction\beta_k, l) : k, l < n \text{ and } c_{\omega\alpha,n} \notin U(t\restriction\beta_k, l)\}$$

and

$$U(t,n) = \varphi(t)(c_{\omega\alpha,n}) \setminus \left(W \cup \bigcup_{m<n} U(t,m)\right)$$

Otherwise, since $\xi_n \notin C_{\omega\alpha}$, there exists a $\tau(\varphi(t))$-compact open neighborhood $U'(t,n)$ of ξ_n such that $U'(t,n) \subseteq \varphi(t)(\xi_n)$ and $U'(t,n)\cap C_{\omega\alpha} = \varnothing$. Let

$$W = \bigcup\{U(t\restriction\beta_k, l) : k, l < n \text{ and } \xi_n, c_{\omega\alpha,n} \notin U(t\restriction\beta_k, l)\}$$

and define

$$U(t,n) = (\varphi(t)(c_{\omega\alpha,n}) \cup U'(t,n)) \setminus \left(W \cup \bigcup_{m<n} U(t,m)\right)$$

It is easy to see that $U(t,n)$ is a $\tau(\varphi(t))$-compact open neighborhood of $c_{\omega\alpha,n}$ satisfying (xi), $\xi_n \in \bigcup_{m\leq n} U(t,m)$ and for every $m \neq n$, $c_{\delta,m} \notin U_{t,n}$. $\psi(t,0)$ and $\psi(t,1)$ are automatically defined.

We shall show that φ and ψ satisfy the required conditions. Let f be a cofinal branch of T. By a standard argument, we can show that $\tau(\varphi(f))$ is a sub-Ostaszewski topology. Suppose that $\alpha \in (0,\omega_1)$ and x is an unbounded subset of $\omega\alpha$ of order type ω which has a $\tau(\varphi(f))$-compact closure. We shall show that $x \subseteq^* \psi(f\restriction\alpha, f(\alpha))$. Suppose not. Then $x \setminus \psi(f\restriction\alpha, f(\alpha)) = x\cap\psi(f\restriction\alpha, 1-f(\alpha))$ is unbounded. Since x has a $\tau(\varphi(f))$-compact closure, so does $x\cap\psi(f\restriction\alpha, 1-f(\alpha))$. Therefore, there exists a $y \subseteq x\cap\psi(f\restriction\alpha, 1-f(\alpha))$ such that there exists a unique element $\gamma \in \mathrm{cl}_{\tau(\varphi(f))}(y)\setminus\omega\alpha$. Then $y \subseteq^* \varphi(f)(\gamma)$. But, by (x), there are at most finitely many $l < \omega$ such that $U(f\restriction\alpha, 2l+1-f(\alpha))\cap\varphi(f)(\gamma) \neq \varnothing$. Since $U(f\restriction\alpha, 2l+1-f(\alpha))$ is bounded in $\omega\alpha$ for every $l < \omega$, it follows that $\psi(f\restriction\alpha, 1-f(\alpha))\cap\varphi(f)(\gamma) =$

$\bigcup_{l<\omega} U(f \restriction \alpha, 2l+1-f(\alpha)) \cap \varphi(f)(\gamma)$ is bounded in $\omega\alpha$. Thus, y is bounded in $\omega\alpha$. Since $y \subseteq x$, $\mathrm{ot}(x) = \omega$, and $\sup(x) = \omega\alpha$, it implies that y is finite and hence $\tau(\varphi(f))$-closed. It contradicts $\gamma \in \mathrm{cl}_{\tau(\varphi(f))}(y) \setminus \omega\alpha$. Thus, we obtained $x \subseteq^* \psi(f \restriction \alpha, f(\alpha))$. □

Let φ and ψ be as in the conclusion of the previous lemma. Suppose that f is a cofinal branch of T and F is an unbounded $\tau(\varphi(f))$-closed set such that every initial segment has a $\tau(\varphi(f))$-compact closure. Fix $\alpha < \omega_1$ such that $F \cap \omega\alpha$ is unbounded in $\omega\alpha$. Let $x \subseteq F \cap \omega\alpha$ be an arbitrary unbounded set. Then $f(\alpha) = i$ if and only if $x \subseteq^* \psi(f \restriction \alpha, i)$. Therefore, we can retrieve the value of $f(\alpha)$ from $F \cap \omega\alpha$ and $f \restriction \alpha$.

3. Deriving ¬ CH

Theorem 3.1. *Suppose that* CH *holds,* $\omega_1 = (\omega_1)^L$*, and* $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)^{L[A]}$ *for some* $A \subseteq \omega_1$*. Then there exists a function* g *which defines a thin topology on* ω_1 *such that*

(i) $(\omega_1, \tau(g))$ *is a locally compact, sub-Ostaszewski space in* $L[A, g]$*, and*
(ii) *there exists no uncountable* $\tau(g)$*-compact subset* F *of* ω_1 *such that every initial segment of* F *has* $\tau(g)$*-compact closure.*

Proof. Suppose not. Let $\mathcal{F}$ be the set of all countable sequence $\langle G_i, g_i : i < j\rangle$ such that there exists a α such that for every $i < j$, G_i is a subset of $\omega\alpha$ and g_i is a function from α into 2. Since $\mathcal{P}(\omega) = (\mathcal{P}(\omega))^{L[A]}$, we have $\mathcal{F} = (\mathcal{F})^{L[A]}$. Since CH holds in $L[A]$, there exists a bijection $\sigma : \mathcal{P}(\omega) \to \mathcal{F}$ in $L[A]$.

Let f be any cofinal branch of T. First we shall inductively construct a subset F_i of ω_1, and a cofinal branch f_i of T for all $i < \omega\omega$.

Apply Lemma 2.1 in $L[A]$ to get φ_0 and ψ_0 satisfying the conclusion. For every $n < \omega$, define $f_n = f$. Let $\tau_n = \tau(\varphi(f_n))$. In V, by assumption, there exists an uncountable τ_n-closed set F_n such that every initial segment of F_n has τ_n-compact closure.

Now suppose that $\langle F_i : i < \omega k\rangle$, and $\langle f_i : i < \omega k\rangle$ has been defined. Let $D_k = \bigcap_{i<\omega k} \lim(F_i) \cap \omega_1$. Note that we can easily code A and $\langle F_i, f_i : i < \omega k\rangle$ by a subset A_k of ω_1 so that for every limit ordinal δ, $A \cap \delta$ and $\langle F_i \cap \delta, f_i \restriction \delta : i < \omega k\rangle$ can be retrieved from $A_k \cap \delta$ and vice versa. Apply Lemma 2.1 in $L[A_k]$ to obtain φ_k and ψ_k. Then if $t \in \mathcal{T}$ and $\alpha = \mathrm{dom}(t)$, then $\varphi_k(t)$, $\psi_k(t, 0)$, and $\psi_k(t, 1)$ are computed from t, $A \cap \delta$, and $\langle F_i \cap \delta, f_i \restriction \delta : i < \omega k\rangle$ in $L[A \cap \delta, \langle F_i \cap \delta, f_i \restriction \delta : i < \omega k\rangle]$. Let $\bar{D}_k = \{\alpha < \omega_1 : \omega\alpha \in D_k\}$. For each $\alpha \in \bar{D}_k$, define $a_{k,\alpha} = \sigma^{-1}(\langle F_i \cap \omega\alpha, f_i \cap \omega\alpha : i < \omega k\rangle)$. For each $n < \omega$, define $f_{\omega k+n}$ by

$$f_{\omega k+n}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \bar{D}_k \text{ and } n \in a_{k,\alpha} \\ 0 & \text{otherwise} \end{cases}$$

Let $\tau_{\omega k+n} = \tau(\varphi_k(f_{\omega k+n}))$. Then in V, there exists an unbounded $\tau_{\omega k+n}$-closed subset $F_{\omega k+n}$ such that every initial segment of $F_{\omega k+n}$ has $\tau(\varphi_k(f_{\omega k+n}))$-compact closure. It finishes the definition of $\langle F_i, f_i : i < \omega\omega\rangle$.

Define $D = \bigcap_{i<\omega\omega} \lim(F_i) \cap \omega_1$. Let $\delta_0 = \min(D)$. Consider $\langle F_i \cap \delta_0, f_i \restriction \delta_0 : i < \omega\omega\rangle$. We shall show that from this sequence, we can retrieve f. Since there are $\aleph_1$-many possibility for this sequence, it implies that there are at most $\aleph_1$-many cofinal branches of T. This is a contradiction.

The following claim suffices.

Claim 1. Let $\delta \in D$ and $\delta' = \min(D \setminus (\delta + 1))$. Set α and α' be so that $\omega\alpha = \delta$ and $\omega\delta' = \delta'$ Then we can compute $\langle F_i \cap \delta', f_i \restriction \alpha' : i < \omega\omega\rangle$ from $\langle F_i \cap \delta, f_i \restriction \alpha : i < \omega\omega\rangle$.

$\vdash$ Fix $k < \omega$. we shall show that we can compute $\beta = \min(\bar{D}_k \setminus (\alpha+1))$, and $\langle F_i \cap \omega\gamma, f_i \restriction \beta : i < \omega\omega\rangle$ from $\langle F_i \cap \delta, f_i \restriction \alpha : i < \omega k\rangle$. It clearly suffices. We can compute $\varphi_k(f_i \restriction \alpha)$. $\psi_k(f_i \restriction \alpha, 0)$, and $\psi_k(f_i \restriction \alpha, 1)$. For every $n < \omega$, let $x_{\omega k+n}$ be any unbounded subset of $F_{\omega k+n} \cap \delta$ of order type ω. We know that $x_{\omega k+n} \subseteq^* \psi(f_{\omega k+n} \restriction \alpha, j)$ if and only if $f_{\omega k+n}(\alpha) = j$. Therefore, we can retrive $f_{\omega k+n}(\alpha)$ for every $n < \omega$ and hence $a_{\omega k+n}$. Recall that $\sigma(x_{\omega k+n}) = \langle F_i \cap \omega\beta, f_i \restriction \beta : i < \omega k\rangle$. Thus, we can compute $\langle F_i \cap \omega\beta, f_i \restriction \beta : i < \omega k\rangle$. $\dashv$ (Claim1)

$\square$(Theorem3.1)

For example, suppose P forces that CH holds and there is no locally compact, sub-Ostaszewski space. Then P kills all such spaces in the intermediate models. However, for at least one such space (ω_1, τ), P adds an unbounded counbounded τ-closed set which has an initial segment with non τ-compact closure. This does not totally rule out even the possibility that the model of Eisworth and Roitman in [1] has no sub-Ostaszewski space because some iterand may accidentally kills a locally compact, sub-Ostaszewski space in such a way. Nonetheless, this explains why sub-Ostaszewski spaces are much harder to deal with than Ostaszewski spaces.

In fact, $\omega_1 = (\omega_1)^{V[A]}$ is not necessary. If V satisfies CH, then there exists a subset A of ω_1, such that $L[A]$ contains all reals. Thus, we can show that in V, there exists a function f on ω_1 which defines a thin topology such that $\tau(f)$ is a sub-Ostaszewski space in an inner model $W \subseteq V$ and there is no unbounded $\tau(f)$-closed subset F whose all initial segment have $\tau(f)$-compact closure.

References

1. Todd Eisworth and Judith Roitman, CH *with no Ostaszewski spaces*, Trans. Amer. Math. Soc. **351** (1999), no. 7, 2675–2693. MR **MR1638230 (2000b:**03182)