THE PRECIPITOUSNESS OF TAIL CLUB GUESSING IDEALS

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Abstract. From a measurable cardinal, we build a model in which the nonstationary ideal on ω_1 is not precipitous, but there is a precipitous tail club guessing ideal on ω_1.

1. Introduction

Club guessing sequences were introduced by Shelah in 80's, for example in [8]. Since then, they were proved to be effective tools to show the results under ZFC. The ideals associated with club guessing sequences are also used in various arguments. There are two types of club guessing sequences, tail club guessing sequences and fully club guessing sequences. In this paper, we shall concentrate on tail club guessing sequences.

When $\overrightarrow{C}$ is a tail club guessing sequence on κ, then we can define the filter $\mathrm{TCG}(\overrightarrow{C})$ on κ associated with $\overrightarrow{C}$, which is called the tail club guessing filter. The definition is essentially due to Shelah. The tail club guessing ideal simply refers to the dual ideal of the tail club guessing filter. When Γ is a property of ideals, we say that a filter F has Γ if and only if its dual ideal has Γ. When F is a filter, $\breve{F}$ denotes the dual ideal of F.

There are several results about the precipitousness of tail club guessing ideals. In [10], Woodin proved that it is consistent relative to the consistency of a Woodin cardinal that NS_{ω_1} is $\aleph_2$-saturated and there exists a tail club guessing ideal $\overrightarrow{C}$ on ω_1 such that $\mathrm{NS}_{\omega_1} = \mathrm{T\breve{C}G}(\overrightarrow{C})$, in particular $\mathrm{TCG}(\overrightarrow{C})$ is precipitous. In [5], the author showed that if we collapse a Woodin cardinal to ω_2 by the Levy collapse, then NS_{ω_1} is precipitous and so is every tail club guessing ideal on ω_1. In the same paper, it was thus asked if it is consistent that NS_{ω_1} is not precipitous but there is a precipitous tail club guessing ideal on ω_1. This is the question we shall answer in this paper. In addition, the model is built from a measurable cardinal. Hence, it also shows that the existence of a precipitous tail club guessing ideal is equiconsistent with the existence of a measurable cardinal.

We follow the standard notations in set theory. Lim stands for the class of limit ordinals. When X and Y are sets of ordinals, we say that X is *almost contained in* Y and write $X \subseteq^* Y$ if and only if there exists a $\zeta < \sup(X)$ such that $X \setminus \zeta \subseteq Y$. When X is a set of ordinals, we define $\mathrm{nacc}(X) = X \setminus \lim(X)$ where $\lim(X)$ denote the set of limit points of X. An ordinal α is *indecomposable* if and only if for every $\beta < \alpha$, $\beta + \alpha = \alpha$. When F is a filter on κ, we say that a subset X of κ is *F-positive*

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if and only if $\kappa \setminus X \notin F$. F^+ denotes the set of all F-positive subsets of κ. We automatically assume that $\dot{x}$ is a name for x.

2. Tail club guessing ideals

The following notions were introduced by Shelah in [8] though he used different terminology.

Definition 2.1. Let κ be an uncountable regular cardinal. We say that a sequence $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \mathrm{Lim} \rangle$ is a *tail club guessing sequence on* κ if and only if

(i) for every $\delta \in \kappa \cap \mathrm{Lim}$, C_δ is an unbounded subset of δ, and
(ii) for every club subset D of κ, there exists a $\delta \in \kappa \cap \mathrm{Lim}$ such that $C_\delta \subseteq^* D$.

We say that $\overrightarrow{C}$ has order type ε if and only if for every $\delta \in (\kappa \setminus \varepsilon) \cap \mathrm{Lim}$, $\mathrm{otp}(C_\delta) = \varepsilon$.

We define the *tail club guessing filter* $\mathrm{TCG}(\overrightarrow{C})$ *associated with* $\overrightarrow{C}$ as the filter on κ generated by the sets of the form $\{\delta \in \kappa \cap \mathrm{Lim} : C_\delta \subseteq^* D\}$ for some club subset D of κ. A tail club guessing ideal is the dual ideal of a tail club guessing filter.

In [8], Shelah showed that $\mathrm{TCG}(\overrightarrow{C})$ is κ-complete and normal for every tail club guessing sequence $\overrightarrow{C}$ on κ. Note that not all club guessing sequences have order types. However, for every tail club guessing sequence $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \mathrm{Lim} \rangle$, either there exists an $X \in \mathrm{TCG}(\overrightarrow{C})$ such that for every $\delta \in X$, $\mathrm{otp}(C_\delta) = \delta$ or there exists an $\varepsilon < \kappa$ such that $\{\delta \in \kappa \cap \mathrm{Lim} : \mathrm{otp}(C_\delta) = \varepsilon\}$ is $\mathrm{TCG}(\overrightarrow{C})$-positive.

The following forcing notion was used in [9] by Shelah.

Definition 2.2. Let $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \mathrm{Lim} \rangle$ be a tail club guessing sequence on an uncountable regular cardinal κ. For every $X \in \mathrm{TCG}(\overrightarrow{C})^+$, we define the *standard forcing* $P(\overrightarrow{C}, X)$ *to shoot a* $\mathrm{TCG}(\overrightarrow{C})$*-measure one set through* X as follows: $p \in P(\overrightarrow{C}, X)$ if and only if p is a closed bounded subset of κ such that for every $\delta \in p \cap \mathrm{Lim}$, if $C_\delta \subseteq^* p$, then $\delta \in X$. $P(\overrightarrow{C}, X)$ is ordered by end-extension.

It is easy to see that $P(\overrightarrow{C}, X)$ forces that $\overrightarrow{C}$ is a tail club guessing sequence on κ and $X \in \mathrm{TCG}(\overrightarrow{C})$. In [6], to investigate this forcing notion, the author defined the following properties of club guessing sequences.

Definition 2.3. Let κ be an uncountable regular cardinal and $\tau : \kappa \to [\kappa]^{<\kappa}$ We say that a subset X of κ is τ*-weakly tight* if and only if for every $\gamma \in \mathrm{nacc}(X)$, $X \cap \gamma \in \tau''\gamma$.

Definition 2.4. Let $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \mathrm{Lim} \rangle$ be a tail club guessing sequence on an uncountable regular cardinal κ.

(i) We say that $\overrightarrow{C}$ is *weakly tight* if and only if there exists a function $\tau : \kappa \to [\kappa]^{<\kappa}$ such that for every $\delta \in \kappa \cap \mathrm{Lim}$, C_δ is τ-weakly tight.
(ii) We say that $\overrightarrow{C}$ is *simple* if and only if for every $\delta \in \kappa \cap \mathrm{Lim}$ and $\gamma \in C_\delta \cap \mathrm{Lim}$, $C_\gamma \setminus C_\delta$ is unbounded in γ.

For example, if $\overrightarrow{C}$ has order type ε, then $\overrightarrow{C}$ is simple. If $\overrightarrow{C}$ has order type ω, then $\overrightarrow{C}$ is also weakly tight. Standard constructions of club guessing sequences often yields simple weakly tight sequences.

We shall review several properties of forcing notions.

Definition 2.5. A sequence of $\langle N_\alpha : \alpha < \eta \rangle$ is called a *tower* if and only if

(i) for every limit $\alpha < \eta$, $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$, and
(ii) for every $\alpha < \eta$, $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$.

Typically, each N_α is a countable elementary submodel of $H(\theta)$ for some large regular cardinal θ.

Definition 2.6. Let $\varepsilon < \omega_1$. A forcing notion P is called *ε-proper* if and only if whenever $\langle N_\alpha : \alpha < \varepsilon \rangle$ is a tower of countable elementary submodels of $H(\theta)$ for some sufficiently large regular cardinal θ with $P \in N_0$, for every $p \in P \cap N_0$, there exists a $q \leq p$ that is (N_α, P)-generic for every $\alpha < \varepsilon$.

We say that P is $<\varepsilon$-proper if and only if P is ε'-proper for every $\varepsilon' < \varepsilon$.

Lemma 2.7 (Shelah [9])**.** *Let $\varepsilon < \omega_1$. Let $\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \leq \eta \rangle$ be a countable-support iteration such that for each $\alpha < \eta$, P_α forces that $\dot{Q}_\alpha$ is ε-proper. Then P_η is ε-proper.*

Definition 2.8. A forcing notion P is called *totally proper* if and only if P is proper and adds no new reals.

Let N be a countable elementary submodel of $H(\theta)$ for some sufficiently large regular cardinal θ with $P \in N$. We say that $p \in P$ is totally (N, P)-generic if and only if p is (N, P)-generic and decides all dense subsets of P lying in N.

Unlike ε-properness, total properness is not preserved by countable support iteration. It was discussed by Shelah in [9].

The following lemma was proved by the author in [6].

Lemma 2.9. *Let $\overrightarrow{C} = \langle C_\delta : \delta \in \omega_1 \cap \mathrm{Lim} \rangle$ be a simple weakly tight tail club guessing sequence on ω_1 and $X \in \mathrm{TCG}(\overrightarrow{C})^+$.*

(i) *$P(\overrightarrow{C}, X)$ is totally proper.*
(ii) *If $\overrightarrow{C}$ has order type ε for some indecomposable ordinal ε, then $P(\overrightarrow{C}, X)$ is $<\varepsilon$-proper.*

The following notion is exactly the same as what was called an outside club guessing sequence by Džamonja and Shelah in [2].

Definition 2.10. Let W be an inner model of V and κ an uncountable regular cardinal in W. Then, we say that a subset C of κ is a *fast club subset of κ over W* if and only if for every club subset D of κ lying in W, $C \subseteq^* D$.

We also use the iteration in the sense of Donder and Fuchs in [1]. They use the system of projections instead of regular embeddings and only consider a sequence of complete Boolean algebras, but our definition is essentially the same in this situation.

Definition 2.11. Let $\langle P_\alpha : \alpha < \eta \rangle$ be a sequence of forcing notions. We say that $\langle P_\alpha : \alpha < \eta \rangle$ is an *iteration* if there exists a system of functions $\langle \sigma_{\beta,\alpha} : \beta < \alpha < \eta \rangle$ such that for every $\beta < \alpha < \eta$, $\sigma_{\beta,\alpha} : P_\beta \to P_\alpha$ is a regular embedding and for every $\gamma < \beta < \alpha < \eta$, $\sigma_{\gamma,\alpha} = \sigma_{\beta,\alpha} \circ \sigma_{\gamma,\beta}$.

By arranging the representation of P_α's, we may assume that $\sigma_{\beta,\alpha}$ is an inclusion map for every $\beta < \alpha < \eta$.

Definition 2.12. Let $\langle P_\alpha : \alpha < \eta \rangle$ be an iteration witnessed by $\langle \sigma_{\beta,\alpha} : \beta < \alpha < \eta \rangle$. The *direct limit* of $\langle P_\alpha : \alpha < \eta \rangle$ is the forcing notion $P = \bigcup_{\alpha<\eta} P_\alpha$. When $\beta < \alpha$, $p \in P_\beta$ and $q \in P_\alpha$, we define $p \leq_P q$ if and only if $\sigma_{\beta,\alpha}(p) \leq_{P_\alpha} q$.

Then, if $G \subseteq P$ is generic, then for every $\alpha < \eta$, $G \cap P_\alpha$ is generic. Notice that if $\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha < \eta \rangle$ is an iteration in Shelah's sense, then $\langle P_\alpha : \alpha < \eta \rangle$ is an iteration in this sense, and the direct limit in both senses coincide.

By the standard argument to find an inner model of a measurable cardinal from a precipitous ideal, we can prove the following lemma.

Lemma 2.13. *Let κ be a measurable cardinal, U a normal measure on κ, and $j : V \to M$ the induced elementary embedding. Suppose $V = L[U]$. Let P be a forcing notion and $G \subseteq P$ generic. Suppose that in $V[G]$, I is a normal precipitous ideal on κ. Let $U_I \subseteq \mathcal{P}(\kappa)/I$ be generic over $V[G]$ and $j_I : V[G] \to N$. Then, $U_I \cap V = U$ and $j_I \upharpoonright V = j$.*

3. A precipitous tail club guessing ideal from a measurable cardinal

This section is devoted to the proof of the following theorem.

Theorem 3.1. *Let κ be a measurable cardinal and $\varepsilon < \kappa$ an indecomposable ordinal. Then, there is a forcing extension in which*

(i) *there exists a tail club guessing sequence $\overrightarrow{C}$ of order type ε such that $\mathrm{TCG}(\overrightarrow{C})$ is precipitous,*
(ii) *no restriction of NS_{ω_1} to any stationary subset of ω_1 is precipitous, and*
(iii) *for every tail club guessing sequence $\overrightarrow{C}'$ of order type $<\varepsilon$, $\mathrm{TCG}(\overrightarrow{C}')$ is not precipitous.*

First let κ be a measurable cardinal and $\varepsilon < \kappa$ an indecomposable ordinal. We shall construct a forcing extension in which there exists a precipitous tail club guessing ideal on ω_1. The construction is eventually used to witness the theorem.

Let U be a normal measure on κ, and $j : V \to M$ the elementary embedding induced by U. Let $P = \mathrm{Coll}(\omega, {<}\kappa)$. Let $G \subseteq P$ be generic over V and $\hat{G} \subseteq j(P)$ generic over M extending G. It is well known that j can be extended to $j_0 : V[G] \to M[\hat{G}]$. Work in $V[G]$. Define an ideal I_0 on ω_1 by: $X \in I_0$ if and only if $\mathbf{1}_{j(P)/G} \Vdash$ '$\kappa \notin \dot{j}_0(X)$'. It is also well known that if we define $\pi_0 : \mathcal{P}(\kappa)/I_0 \to \mathcal{B}(j(P)/G)$ by $\pi_0(X) = [\![\kappa \in \dot{j}_0(X)]\!]$, then π_0 is a dense embedding and hence $j(P)/G \simeq \mathcal{P}(\omega_1)/I_0$. Note $(M[G])^{\aleph_1} \cap V[G] \subseteq M[G]$.

In $V[G]$, we shall define a countable support iteration $\langle Q_\alpha, \dot{R}_\beta : \beta < \alpha \leq \omega_2 \rangle$ by induction so that for every $\alpha < \omega_2$, Q_α forces that $|\dot{R}_\alpha| = \aleph_1$ and $\dot{R}_\alpha$ is ${<}\varepsilon$-proper and totally proper. By a standard argument, we can show that Q_α has a dense subset of size $\aleph_1$. By Lemma 2.7, it follows that for every $\beta < \alpha < \omega_2$ and generic filter $H_\beta \subseteq Q_\beta$ over $V[G]$, Q_α/H_β is ${<}\varepsilon$-proper. In addition, we will show that Q_α is totally proper. We shall also define a Q_α-name $\dot{I}_\alpha$ for an ideal on ω_1. Let $\langle \dot{X}_\alpha : \alpha < \omega_2 \rangle$ be a bookkeeping of all subsets of ω_1 in the extension of $V[G]$ by Q_{κ^+} so that every subset appears unboundedly many times. Since CH holds in $V[G]$, we can pick a bijection $\tau : \omega_1 \to [\omega_1]^{\aleph_0}$. Since Q_α is totally proper for every $\alpha < \omega_2$, τ remains a bijection from ω_1 onto $[\omega_1]^{\aleph_0}$ in the extension of $V[G]$ by Q_α.

At the zero-th stage, let R_0 be the set of all functions r such that $\mathrm{dom}(r) = \delta \cap \mathrm{Lim}$ for some ordinal $\delta < \omega_1$ and for every $\gamma \in \delta \cap \mathrm{Lim}$, $r(\gamma)$ is an unbounded subset of γ, if $\gamma \geq \varepsilon$, then $\mathrm{otp}(r(\gamma)) = \varepsilon$, and for every $\xi \in \mathrm{nacc}(r(\gamma))$, $r(\gamma) \cap \xi \in \tau''\xi$. R_0 is ordered by extension. If $H_0 \subseteq R_0$ is generic over $V[G]$, for every $\delta \in \omega_1 \cap \mathrm{Lim}$, let $C_\delta = r(\delta)$ for some (all) $r \in H_0$ with $\delta \in \mathrm{dom}(r)$. Define $\overrightarrow{C} = \langle C_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$. It is easy to see that $\overrightarrow{C}$ is a simple weakly tight tail club guessing sequence on ω_1 in $V[G][H_0]$. It automatically defines Q_1.

Suppose that we have defined Q_α for some $\alpha \in [1, \kappa^+)$. Let $H_\alpha \subseteq Q_\alpha$ be generic over $V[G]$. Since Q_α has a dense subset of size $\aleph_1$ and $(M[G])^{\omega_1} \cap V[G] \subseteq M[G]$, every Q_α-name for a subset of $\aleph_1$ has an equivalent name lying in $M[G]$.

Work in $V[G][H_\alpha]$. Define I_α by: whenever $\hat{G} \subseteq j(P)$ is generic over M extending G with $H_\alpha \in M[\hat{G}]$ and $\hat{H}_\alpha$ is generic over $M[\hat{G}]$ so that ${j_0}^{-1}\hat{H}_\alpha = H_\alpha$ and $j_0(C)_\kappa$ is a fast club subset of κ over $V[G][H_\alpha]$, we have $\kappa \notin j_\alpha(X)$. Here, $j_\alpha : V[G][H_\alpha] \to M[\hat{G}][\hat{H}_\alpha]$ is the elementary embedding defined by: for every $x \in V[G][H_\alpha]$, if $\dot{x}$ is a Q_α-name for x in $V[G]$, then $j_\alpha(x) = j_0(\dot{x})^{\hat{H}_\alpha}$. This is a highly meta-mathematical definition, but we can modify it into the expression inside $V[G][H_\alpha]$. For simplicity, we let C_κ denote $j_0(C)_\kappa$. If $X_\alpha \notin I_\alpha$, then let R_α be the trivial forcing notion. If $X_\alpha \in I_\alpha$, then let R_α be the standard forcing to shoot a $\mathrm{TCG}(\overrightarrow{C})$-measure one set through $\kappa \setminus X_\alpha$. This completes the definition of $\langle Q_\alpha, \dot{R}_\beta : \beta < \alpha \leq \omega_2\rangle$ and $\langle \dot{I}_\alpha : \alpha < \omega_2\rangle$.

Let $Q = Q_{\omega_2}$. We shall show that in $V[G]$, Q forces that $\mathrm{TCG}(\overrightarrow{C})$ is precipitous.

Lemma 3.1. *Let $\alpha < \kappa^+$. Let $\hat{G} \subseteq j(P)$ be generic over M extending G such that there exists a generic filter $H_\alpha \subseteq Q_\alpha$ over $V[G]$ lying in $M[\hat{G}]$. Then, there exists an unbounded subset $C \in M[\hat{G}]$ of κ such that $\mathrm{otp}(C) = \varepsilon$, C is τ-weakly tight, and C is a fast club subset of κ over $V[G]$.*

Proof. Note $\left|\mathcal{P}(\kappa)^{V[G][H_\alpha]}\right| = \aleph_0$ in $M[\hat{G}]$. So, we can enumerate all club subsets of κ lying in $V[G][H_\alpha]$ as $\langle E_n : n < \omega\rangle$. Now, it is easy to build a C witnessing the claim. □

Lemma 3.2. *Let $\beta < \alpha < \omega_2$. Let $H_\beta \subseteq Q_\beta$ be generic. Suppose that $\hat{G} \subseteq j(P)$ is generic over M extending G with $H_\beta \in M[\hat{G}]$ and $C \in M[\hat{G}]$ is a τ-weakly tight fast club subset of κ over $V[G][H_\beta]$ of order type ε. Then, for every $q \in Q_\alpha$ with $q \restriction \beta \in H_\beta$, there exists a generic filter $H_\alpha \subseteq Q_\alpha$ over $V[G]$ such that $H_\alpha \in M[\hat{G}]$, $H_\beta = H_\alpha \cap Q_\beta$, $q \in H_\alpha$, and C is a fast club subset of κ over $V[G][H_\alpha]$.*

Proof. In $M[\hat{G}]$, $\left|\mathcal{P}(Q_\alpha)^{V[G]}\right| = \aleph_0$. So, we can pick an enumeration $\langle \mathcal{D}_n : n < \omega\rangle \in M[\hat{G}]$ of all dense open subset of Q_α/H_β lying in $V[G][H_\beta]$.

Work in $M[\hat{G}]$. Let θ be a sufficiently large regular cardinal. We shall construct a sequence $\langle N_{n,\xi} : n < \omega$ and $\xi < \omega_1\rangle$ so that for every $n < \omega$, $\langle N_{n,\xi} : \xi < \omega_1\rangle \in M[G][H_\beta]$. In $M[G][H_\beta]$, pick a tower $\langle N_{0,\xi} : \xi < \omega_1\rangle$ of countable elementary submodels of $H(\theta)^{M[G][H_\beta]}$ with $Q_\alpha/H_\beta, q, \mathcal{D}_0 \in N_{0,0}$. Suppose that we have defined $\langle N_{n,\xi} : \xi < \omega_1\rangle$. In $M[G][H_\beta]$, pick a tower $\langle N_{n+1,\xi} : \xi < \omega_1\rangle$ of countable elementary submodels of $H(\theta)^{M[G][H_\beta]}$ such that $\mathcal{D}_{n+1} \in N_{n+1,0}$ and for every $\xi < \omega_1$, $N_{n,\xi} \subseteq N_{n+1,\xi}$. For each $n < \omega$, define $E_n = \{\xi < \omega_1 : N_{n,\xi} \cap \omega_1 = \xi\}$. Then, E_n is a club subset of ω_1 lying in $M[G][H_\beta]$. By assumption, $C \subseteq^* E_n$. Let $\zeta_n \in E_n$ be a successor ordinal so that $C \setminus \zeta_n \subseteq E_n$. Without loss of generality, we may assume that $\langle \zeta_n : n < \omega\rangle$ is increasing.

Keep working in $M[\hat{G}]$. We shall build a decreasing sequence $\langle q_n : n < \omega\rangle$ in Q_α/H_β as follows. Let $q_0 = q$. Suppose that we have defined q_n so that $q_n \in N_{n,\zeta_n}$.

Since $\mathrm{otp}(C) = \varepsilon$, $C \cap [\zeta_n, \zeta_{n+1})$ has order type $<\varepsilon$. Since Q_α/H_β is $<\varepsilon$-proper, there exists a $q_{n+1} \leq q_n$ such that $q_{n+1} \in \mathcal{D}_n \cap N_{n,\zeta_{n+1}}$ and for every $\gamma \in C \cap [\zeta_n, \zeta_{n+1})$, q_{n+1} is $(N_{n,\gamma}, Q_\alpha/H_\beta)$-generic. Notice that $q_{n+1} \in N_{n,\zeta_{n+1}} \subseteq N_{n+1,\zeta_{n+1}}$. Define $H_{\beta,\alpha}$ be the filter generated by $\{q_n : n < \omega\}$. Let H_α be defined by $q' \in H_\alpha$ if and only if $q' \restriction \beta \in H_\beta$ and $q' \restriction [\beta, \alpha)^{M[G][H_\beta]} \in H_{\beta,\alpha}$. Then, it is easy to see that H_α satisfies the desired conditions. □

For each $\alpha < \kappa^+$, we say that $q \in Q_\alpha$ is a *flat condition of height* ζ if and only if for every $\beta \in [1, \alpha)$, $q \restriction \beta$ decides $q(\beta)$ and either $q(\beta) = \varnothing$ or $\max(q(\beta)) = \zeta$. For each $\alpha < \kappa^+$ and $\zeta < \kappa$, let $\mathcal{D}_{\alpha,\zeta}$ be the set of all flat conditions in Q_α of height $\geq\zeta$.

Lemma 3.3. *The following hold for every* $\alpha < \omega_2$.

(i) Q_α *is totally proper.*
(ii) *For every* $\zeta < \omega_1$, $\mathcal{D}_{\alpha,\zeta}$ *is dense in* Q_α.
(iii) *Suppose that* $\hat{G} \subseteq j(P)$ *is generic over* M *extending* G *and* $H_\alpha \subseteq Q_\alpha$ *is generic over* $V[G]$. *Let* $C \in M[\hat{G}]$ *be a fast club subset of* κ *over* $V[G][H_\alpha]$ *of order type* ε. *In* $M[\hat{G}]$, *define* $m'_\alpha \in j_0(Q_\alpha)$ *by for every* $\beta < j(\alpha)$,

$$m'_\alpha(\beta) = \begin{cases} \overrightarrow{C} \cup \{\langle \kappa, C\rangle\} & \text{if } \beta = 0 \\ \varnothing & \text{if } \beta \notin j''\alpha \\ \varnothing & \text{if } \beta = j(\bar{\beta}) \text{ and } X_{\bar{\beta}} \notin I_{\bar{\beta}} \\ D_{\bar{\beta}} \cup \{\kappa\} & \text{if } \beta = j(\bar{\beta}) \text{ and } X_{\bar{\beta}} \in I_{\bar{\beta}} \end{cases}$$

Then, for every $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ *that is generic over* $M[\hat{G}]$ *with* $M[\hat{G}][\hat{H}_\alpha] \models$ '$C_\kappa = C$', ${j_0}^{-1}\hat{H}_\alpha = H_\alpha$ *if and only if* $m'_\alpha \in \hat{H}_\alpha$.
(iv) Q_α *forces that* $\dot{I}_\alpha$ *is non-trivial.*

Proof. Go by induction on α. Suppose that (i), (ii), (iii) and (iv) hold for every $\beta < \alpha$.

We already know that Q_α is proper. First, we shall show (i) and (ii). Let $q \in Q_\alpha$, $\zeta < \omega_1$, and $\dot{x}$ be a Q_α-name for a subset of ω. Without loss of generality, we may assume that $\dot{x} \in M$. Let $\hat{G} \subseteq j(P)$ be generic over M extending G. Work in $M[\hat{G}]$. Since $\left|\mathcal{P}(Q_\alpha)^{V[G]}\right| = \aleph_0$, we can find an enumeration $\langle \mathcal{E}_n : n < \omega\rangle$ of all open dense subsets of Q_α lying in $V[G]$. Also pick an increasing cofinal sequence $\langle \alpha_n : n < \omega\rangle$ in α and an increasing cofinal sequence $\langle \kappa_n : n < \omega\rangle$ in κ. It is easy to build a decreasing sequence $\langle q_n : n < \omega\rangle$ so that $q_0 = q$, $q_{n+1} \in \mathcal{E}_n$, $q_{n+1} \restriction \alpha_n \in \mathcal{D}_{\alpha_n,\kappa_n}$, and q_{n+1} decides $n \in \dot{x}$.

Let $H_\alpha \subseteq Q_\alpha$ be the filter generated by $\{q_n : n < \omega\}$. Then, clearly H_α is generic over $V[G]$. Let $\overrightarrow{C} = \langle C_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ be the sequence added at the zero-th stage and D_β the club subset of ω_1 added at the β-th stage. Let $\bar{x} = \{n < \omega : q_{n+1} \Vdash \text{'} n \in \dot{x}\text{'}\}$. Let $C \in M[\hat{G}]$ be a τ-weakly tight fast club subset of κ over $V[G][H_\alpha]$ of order type ε. Define m'_α as in (iii).

Claim 3.3.1. $m'_\alpha \in j_0(Q_\alpha)$.

Proof. We have $\mathrm{supp}(m'_\alpha) \subseteq j''\alpha$, which is countable in $M[\hat{G}]$. Thus, it suffices to show that $m'_\alpha \restriction \beta \Vdash$ '$m'_\alpha(\beta) \in j_0(\dot{R})_\beta$' for every $\beta < j(\alpha)$. If $\beta \notin j''\alpha$, then this is trivial. Suppose $\beta < \alpha$ and show that $m'_\alpha \restriction j(\beta) \Vdash$ '$m'_\alpha(j(\beta)) \in j_0(\dot{R}_\beta)$'. If either $\beta = 0$ or $X_\beta \notin I_\beta$, this is again trivial. Assume that $\beta > 0$ and $X_\beta \in I_\beta$. Let

$\hat{H}_\beta \subseteq j_0(Q_\beta)$ be generic over $M[\hat{G}]$ with $m'_\alpha \restriction j(\beta) \in \hat{H}_\beta$. Then by (iii) applied to β, we have ${j_0}^{-1}\hat{H}_\beta = H_\alpha \cap Q_\beta$. By the definition of I_β, we have $\kappa \notin j_\beta(X_\beta)$. We also have $j_\beta(X_\beta) \cap \kappa = X_\beta$. Therefore, $m'_\alpha(j(\beta)) = D_\beta \cup \{\kappa\} \in j_\beta(R_\beta)$. Since this holds for arbitrary $\hat{H}_\beta$, $m'_\alpha \restriction j(\beta) \Vdash$ '$m'_\alpha(j(\beta)) \in j_0(\dot{R}_\beta)$'. □

Claim 3.3.2. For every $n < \omega$, $m'_\alpha \leq j_0(q_n)$. In particular, $m'_\alpha \leq j_0(q)$

Proof. Since $\mathrm{supp}(m'_\alpha) \subseteq j''\alpha$, it suffices to show that for all $k \in (n, \omega)$, $m'_\alpha \restriction j(\alpha_k) \leq j_0(q_n \restriction \alpha_k)$. By definition, $q_{k+1} \restriction \alpha_k \in \mathcal{D}_{\alpha_k,\kappa_k}$, in particular $q_{k+1} \restriction \alpha_k$ is a flat condition of height ζ' for some $\zeta' < \kappa$. Thus, $j_0(q_{k+1} \restriction \alpha_k)$ is a flat condition of height ζ' with $\mathrm{supp}(j_0(q_{k+1} \restriction \alpha_k)) \subseteq j''\alpha$. Now it is easy to see $m'_\alpha \restriction j(\alpha_k) \leq j_0(q_{k+1} \restriction \alpha_k) \leq j_0(q_n \restriction \alpha_k)$. □

Claim 3.3.3. $m'_\alpha \Vdash$ '$j_0(\dot{x}) = \bar{x}$'.

Proof. Let $n < \omega$. Recall that q_{n+1} decides $n \in \dot{x}$. If $n \in \bar{x}$, then we have $q_{n+1} \Vdash$ '$n \in \dot{x}$' and hence $j_0(q_{n+1}) \Vdash$ '$n \in j_0(\dot{x})$'. Hence, $m'_\alpha \Vdash$ '$n \in j_0(\dot{x})$'. By the same argument, if $n \notin \bar{x}$, we have $m'_\alpha \Vdash$ '$n \notin j_0(\dot{x})$'. □

Therefore, in $M[\hat{G}]$, there exists an $m'_\alpha \leq j_0(q)$ such that $m'_\alpha \in j_0(\mathcal{D}_{\alpha,\zeta})$ and $m'_\alpha \Vdash$ '$j_0(\dot{x}) = \bar{x}$' for some $\bar{x} \in M[\hat{G}]$. Since $j_0 : V[G] \to M[\hat{G}]$ is an elementary embedding, it shows that in $V[G]$, there exists a $q' \leq q$ such that $q' \in \mathcal{D}_{\alpha,\zeta}$ and $q' \Vdash$ '$\dot{x} = \bar{x}$' for some $x \in V[G]$. Therefore, Q_α adds no new reals and $\mathcal{D}_{\alpha,\zeta}$ is dense in Q_α.

Since the set of flat conditions is dense in Q_α, (iii) can be easily seen.

To see (iv), suppose that for some $q \in Q_\alpha$, $q \Vdash$'$\kappa \in \dot{I}_\alpha$'. Let $\hat{G} \subseteq j(P)$ be generic over M extending G. In $M[\hat{G}]$, we can find an $H_\alpha \subseteq Q_\alpha$ generic over $V[G]$ with $q \in H_\alpha$. Let $C \in M[\hat{G}]$ be a τ-weakly tight fast club subset of κ over $V[G][H_\alpha]$ of order type ε. Let m'_α be defined as in (iii). Let $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ be generic over $M[\hat{G}]$ with $m'_\alpha \in \hat{H}_\alpha$. Since $q \Vdash$'$\kappa \in \dot{I}_\alpha$', we have $\kappa \notin j_\alpha(\kappa)$. This is a contradiction. □

Lemma 3.4. *Let $\alpha < \omega_2$ and $H_\alpha \subseteq Q_\alpha$ be generic over $V[G]$. Then, in $V[G][H_\alpha]$, $\breve{\mathrm{TCG}}(\overrightarrow{C}) \subseteq I_\alpha$, and hence $\overrightarrow{C}$ is a tail club guessing sequence on ω_1.*

Proof. Work in $V[G][H_\alpha]$. Suppose that there exists a club subset D of κ such that $X := \{\delta \in \omega_1 \cap \mathrm{Lim} : C_\delta \not\subseteq^* D\} \notin I_\alpha$. By the definition of I_α, there exist a generic filter $\hat{G} \subseteq j(P)$ over M extending G with $H_\alpha \in M[\hat{G}]$ and a generic filter $\hat{H}_\alpha \subseteq j(Q_\alpha)$ over $M[\hat{G}]$ such that ${j_0}^{-1}\hat{H}_\alpha = H_\alpha$, C_κ is a fast club over $V[G][H_\alpha]$, and $\kappa \in j_\alpha(X)$. Since $\kappa \in j_\alpha(X)$, we have $C_\kappa \not\subseteq^* j_\alpha(D)$, which implies $C_\kappa \not\subseteq^* D$. Since C_κ is a fast club over $V[G][H_\alpha]$, this is a contradiction. □

Work in $M[\hat{G}]$ where $\hat{G} \subseteq j(P)$ is generic over M extending G. For every $\alpha < \omega_2$, let $m_\alpha \in j_0(Q_\alpha)$ be the truth value of '${j_0}^{-1}\dot{\hat{H}}_\alpha$ is generic over $V[G]$ and $\dot{C}_\kappa$ is a fast club over $V[G][H_\alpha]$'. Let $\dot{m}_\alpha$ be a $j(P)/G$-name for m_α.

Suppose that $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ is generic over $M[\hat{G}]$ with $m_\alpha \in \hat{H}_\alpha$. Let $H_\alpha = {j_0}^{-1}\hat{H}_\alpha$. Since $m_\alpha \in \hat{H}_\alpha$, H_α is generic over $V[G]$. So, we can define j_α as above.

Lemma 3.5. *In $M[G]$, for every $\beta < \alpha < \kappa^+$, $j(P)/G$ forces that $\dot{m}_\beta = \dot{m}_\alpha \restriction j(\beta)$.*

Proof. Let $\hat{G} \subseteq j(P)$ be generic over M extending G and work in $M[\hat{G}]$.

First we shall show that $m_\alpha \restriction \beta \leq m_\beta$. Let $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ be generic over $M[\hat{G}]$ with $m_\alpha \in \hat{H}_\alpha$. Define $H_\alpha = {j_0}^{-1}\hat{H}_\alpha$. By the definition of m_α, H_α is generic over

$V[G]$. $\hat{H}_\beta = \hat{H}_\alpha \cap j_0(Q_\beta)$ is a generic filter of $j_0(Q_\beta)$ over $M[\hat{G}]$. Let $H_\beta = {j_0}^{-1}\hat{H}_\beta$. Then, it is easy to see $H_\beta = H_\alpha \cap Q_\beta$. So, $H_\beta \subseteq Q_\beta$ is generic over $V[G]$. Since C_κ is a fast club over $V[G][H_\alpha]$, it is a fast club over $V[G][H_\beta]$, too. By definition, we have $m_\beta \in \hat{H}_\beta$. Since this holds for arbitrary generic $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ over $M[\hat{G}]$ with $m_\alpha \in \hat{H}_\alpha$, we have $m_\alpha \restriction j(\beta) \leq m_\beta$.

Then we shall show that $m_\beta \leq m_\alpha \restriction j(\beta)$. Let $\hat{H}_\beta \subseteq j_0(Q_\beta)$ be generic over $M[\hat{G}]$ with $m_\beta \in \hat{H}_\beta$ and define $H_\beta = {j_0}^{-1}\hat{H}_\beta$. By Lemma 3.2, there exists a generic filter $H_\alpha \subseteq Q_\alpha$ over $V[G]$ such that $H_\alpha \in M[\hat{G}]$, $H_\beta = H_\alpha \cap Q_\beta$, and C_κ is a fast club over $V[G][H_\alpha]$. Let $m'_\alpha \in j_0(Q_\alpha)$ be defined as in (ii) of Lemma 3.3 where C_κ is used for the fast club. Then, we have $m'_\alpha \restriction j(\beta) \in \hat{H}_\beta$. Let $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ be generic over $M[\hat{G}]$ extending $\hat{H}_\beta$ with $m'_\alpha \in \hat{H}_\alpha$. Then, it is easy to see $m_\alpha \in \hat{H}_\alpha$. Hence, we get $m_\beta \leq m_\alpha \restriction j(\beta)$. □

Suppose that $\hat{G} \subseteq j(P)$ is generic over M extending G and $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ is generic over $M[\hat{G}]$ with $m_\alpha \in \hat{H}_\alpha$. For every $\beta < \alpha$, define $\hat{H}_\beta = \hat{H}_\alpha \cap j_0(Q_\beta)$ and $H_\beta = {j_0}^{-1}\hat{H}_\beta$. Since $m_\beta = m_\alpha \restriction j(\beta) \in \hat{H}_\beta$, we can define $j_\beta : V[G][H_\beta] \to M[\hat{G}][\hat{H}_\beta]$ as for j_α. It is easy to see that j_β is an elementary embedding and $j_\beta = j_\alpha \restriction V[G][H_\beta]$.

The following lemma is a consequence of the Duality Theorem proved by Foreman in [3]. See [4] for information about the Duality Theorem.

Lemma 3.6. *In $V[G]$, for every $\alpha < \kappa^+$, $Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) \simeq (j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha)$*

Proof. We shall define a function $\pi_\alpha : Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) \to \mathcal{B}((j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha))$. For every $\langle q, \dot{X}\rangle \in Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha)$, let $\pi_\alpha(\langle q, \dot{X}\rangle)$ be the truth value of the following statement taken in $M[G]$: there exists an $H_\alpha \subseteq Q_\alpha$ such that $H_\alpha \in M[\dot{\hat{G}}]$, H_α is generic over $V[G]$, $q \in H_\alpha$, $\dot{C}_\kappa$ is a fast club over $V[G][H_\alpha]$, $\dot{j}_0^{-1}\dot{\hat{H}}_\alpha = H_\alpha$, and $\kappa \in j_\alpha(\dot{X}^{H_\alpha})$.

We shall show that π_α is a dense embedding. It is easy to see that π_α is well-defined and order-preserving. To see that π_α preserves incompatibility, let $\langle q, \dot{X}\rangle$ and $\langle q', \dot{X}'\rangle$ be incompatible elements in $Q_\alpha * (\mathcal{P}(\kappa)/\dot{I}_\alpha)$. It follows that whenever q'' is a common extension of q and q', $q'' \Vdash$ '$\dot{X} \cap \dot{X}' \in \dot{I}_\alpha$' Suppose that $\pi_\alpha(\langle q, \dot{X}\rangle)$ and $\pi_\alpha(\langle q', \dot{X}'\rangle)$ are compatible in $j(P * \dot{Q}_\alpha)$. Let $\hat{G} * \hat{H}_\alpha$ be a generic filter of $j(P * \dot{Q}_\alpha)$ with $\pi_\alpha(\langle q, \dot{X}\rangle), \pi_\alpha(\langle q', \dot{X}'\rangle) \in \hat{G} * \hat{H}_\alpha$. Let $H_\alpha = {j_0}^{-1}\hat{H}_\alpha$. Then, both q and q' belong to H_α, and hence we have $X \cap X' \in I_\alpha$. But we also have $\kappa \in j_\alpha(X) \cap j_\alpha(X') = j_\alpha(X \cap X')$. This contradicts $X \cap X' \in I_\alpha$.

We shall show that the image of π_α is dense in $\mathcal{B}((j(P)/G) * (\dot{j}_0(Q_\alpha)/\dot{m}_\alpha))$. Let $\langle p, \dot{r}\rangle \in (j(P)/G) * (\dot{j}_0(Q_\alpha)/\dot{m}_\alpha)$. Since $\mathcal{P}(\kappa)/I_0$ is densely embedded into $\mathcal{B}(j(P)/G)$ by the mapping $X \mapsto [\![\kappa \in \dot{j}_0(X)]\!]$, we can find an $X \in \mathcal{P}(\omega_1)/I_0$ such that $[\![\kappa \in \dot{j}_0(X)]\!] \leq p$. Notice that $\dot{r}$ is represented by a function $f_{\dot{r}}$ with domain ω_1 such that for every $\xi < \omega_1$, $f_{\dot{r}}(\xi)$ is a P-name for an element of $\dot{Q}_\alpha$. Define a function $f_r : \omega_1 \to Q_\alpha$ by $f_r(\xi) = (f_{\dot{r}}(\xi))^G$. Define a Q_α-name $\dot{Y}$ for a subset of κ so that for every $\xi \in X$, $[\![\xi \in \dot{Y}]\!] = f_r(\xi)$ and for every $\xi \in \kappa \setminus X$, $\mathbf{1}_{Q_\alpha} \Vdash$ '$\xi \notin \dot{Y}$'.

We claim that $[\![\dot{Y} \notin \dot{I}_\alpha]\!] \neq 0$. Let $\hat{G} \subseteq j(P)$ be generic over M extending G with $M[\hat{G}] \models$ '$\kappa \in j_0(X)$' and $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ generic over $M[\hat{G}]$ with $m_\alpha, r \in \hat{H}_\alpha$. Let $H_\alpha = {j_0}^{-1}\hat{H}_\alpha$. By the definition of m_α, $H_\alpha \subseteq Q_\alpha$ is generic over $V[G]$ and C_κ is a fast club over $V[G][H_\alpha]$. Since $V[G] \models$ '$\forall \xi \in X(f_r(\xi) \Vdash$ '$\xi \in \dot{Y}$')', we have

$j_0(f_r)(\kappa) = r \Vdash$ '$\kappa \in \dot{j}_0(\dot{Y})$'. Therefore, $\kappa \in j_\alpha(Y)$ in $M[\hat{G}][\hat{H}_\alpha]$. It follows that $Y \notin I_\alpha$. Thus, there exists a $q \in H_\alpha$ such that $q \Vdash$ '$\dot{Y} \notin \dot{I}_\alpha$'. Therefore, $\langle q, \dot{Y}\rangle \in Q_\alpha * (\mathcal{P}(\kappa)/\dot{I}_\alpha)$.

We shall show that $\pi_\alpha(\langle q, \dot{Y}\rangle) \leq \langle p, \dot{r}\rangle$. Let $\hat{G} * \hat{H}_\alpha \subseteq (j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha)$ be generic over $M[G]$ with $\pi_\alpha(\langle q, \dot{Y}\rangle) \in \hat{G} * \hat{H}_\alpha$. It suffices to show $\langle p, \dot{r}\rangle \in \hat{G} * \hat{H}_\alpha$. Since $\pi_\alpha(\langle q, \dot{Y}\rangle) \in \hat{G} * \hat{H}_\alpha$, there exists an $H_\alpha \subseteq Q_\alpha$ such that $H_\alpha \in M[\hat{G}]$, H_α is generic over $V[G]$, $q \in H_\alpha$, C_κ is a fast club over $V[G][H_\alpha]$, $H_\alpha = {j_0}^{-1}\hat{H}_\alpha$, and $\kappa \in j_\alpha(\dot{Y}^{H_\alpha})$. Let $Y = \dot{Y}^{H_\alpha}$. By the definition of Y, we have $Y \subseteq X$. Hence $\kappa \in j_\alpha(X)$. Since $[\![\kappa \in \dot{j}_0(X)]\!] \leq p$, we have $p \in \hat{G}$. Note that since $[\![\xi \in \dot{Y}]\!]_{Q_\alpha} = r_\xi$ for every $\xi < \omega_1$, we have $[\![\kappa \in \dot{j}_\alpha(\dot{Y})]\!]_{j(Q_\alpha)} = r$. Since $\kappa \in j_\alpha(Y)$, we have $r \in \hat{H}_\alpha$. □

Lemma 3.7. *Let $\beta < \alpha < \kappa^+$. Suppose that $H_\alpha \subseteq Q_\alpha$ is generic over $V[G]$. Then, in $V[G][H_\alpha]$, $I_\beta = I_\alpha \cap V[G][H_\alpha \cap Q_\beta]$.*

Proof. Let $X \in I_\beta$. To show that $X \in I_\alpha$, let $\hat{G} \subseteq j(P)$ be generic over M extending G with $H_\alpha \in V[\hat{G}]$ and $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ generic over $V[\hat{G}]$ such that ${j_0}^{-1}\hat{H}_\alpha = H_\alpha$ and C_κ is a fast club over $V[G][H_\alpha]$. Since $X \in I_\beta$, we have $\kappa \notin j_\beta(X)$. But we have $j_\beta = j_\alpha \upharpoonright V[G][H_\alpha \cap Q_\beta]$. So, $\kappa \notin j_\alpha(X)$ and hence $X \in I_\alpha$.

To see the converse, suppose that for some $q \in Q_\alpha$ and Q_β-name $\dot{X}$ for a subset of ω_1, $q \Vdash$ '$\dot{X} \in \dot{I}_\alpha \setminus \dot{I}_\beta$'. Let $H_\beta \subseteq Q_\beta$ be generic with $q \upharpoonright \beta \in H_\beta$. Then, we have $X \notin I_\beta$. By definition, there exist a generic filter $\hat{G} \subseteq j(P)$ over M extending G with $H_\beta \in M[\hat{G}]$ and a generic filter $\hat{H}_\beta \subseteq j_0(Q_\beta)$ over $V[\hat{G}]$ such that ${j_0}^{-1}\hat{H}_\beta = H_\beta$, C_κ is a fast club over $V[G][H_\beta]$, and $\kappa \in j_\beta(X)$. By Lemma 3.2, there exists a generic filter $H_\alpha \subseteq Q_\alpha$ over $V[G]$ such that $H_\alpha \in V[\hat{G}]$, $H_\beta = H_\alpha \cap Q_\beta$, $q \in H_\alpha$, and C_κ is a fast club over $V[G][H_\alpha]$. Let $m'_\alpha \in j_0(Q_\alpha)$ be defined as in (iii) of Lemma 3.3 where $C = C_\kappa$. Then, $m'_\alpha \upharpoonright j(\beta) \in \hat{H}_\beta$. Let $\hat{H}_\alpha \subseteq j_0(Q_\alpha)$ be generic over $M[\hat{G}]$ extending $\hat{H}_\beta$ with $m'_\alpha \in \hat{H}_\alpha$. Then, ${j_0}^{-1}\hat{H}_\alpha = H_\alpha$. Since $q \in H_\alpha$, we have $X \in I_\alpha$. Therefore, $\kappa \notin j_\alpha(X)$. This is a contradiction since $j_\beta = j_\alpha \upharpoonright V[G][H_\beta]$. □

Lemma 3.8. *In $V[G]$, for every $\beta < \alpha < \omega_2$, $Q_\beta * (\mathcal{P}(\omega_1)/\dot{I}_\beta)$ is regularly embedded into $Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha)$ by the identity mapping.*

Proof. This holds since the following diagram commutes.

$$\begin{array}{ccc} Q_\beta * (\mathcal{P}(\omega_1)/\dot{I}_\beta) & \xrightarrow{\pi_\beta} & (j(P)/G) * (\dot{j}_0(\dot{Q}_\beta)/\dot{m}_\beta) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) & \xrightarrow{\pi_\alpha} & (j(P)/G) * (\dot{j}_0(\dot{Q}_\alpha)/\dot{m}_\alpha) \end{array}$$

□

Let $\dot{I}$ be a Q-name for $\bigcup_{\alpha<\kappa^+} \dot{I}_\alpha$. The following lemma is an easy consequence of Lemma 3.7.

Lemma 3.9. *In $V[G]$, $Q * (\mathcal{P}(\omega_1)/\dot{I})$ is the direct limit of $\langle Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha) : \alpha < \omega_2\rangle$.*

Lemma 3.10. *In $V[G]$, for every $\alpha < \kappa^+$, let $H_\alpha \subseteq Q_\alpha$ be generic over $V[G]$. Then, in $V[G][H_\alpha]$, I_α is precipitous. Moreover, suppose that $U_\alpha \subseteq \mathcal{P}(\omega_1)/I_\alpha$ is generic over $V[G][H_\alpha]$ and $k_\alpha : V[G][H_\alpha] \to N \subseteq V[G][H_\alpha][U_\alpha]$ is the induced* *generic elementary embedding. Let* $\hat{G} * \hat{H}_\alpha = \pi_\alpha(H_\alpha * U_\alpha)$*. Let* $j_\alpha = \dot{j}_\alpha^{\hat{G}*\hat{H}_\alpha}$*. Then,* $N = M[\hat{G}][\hat{H}_\alpha]$ *and* $k_\alpha = j_\alpha$*.*

Proof. First, we claim that for every $X \in U_\alpha$, we have $\kappa \in j_\alpha(X)$. Let $\dot{X}$ be a Q_α-name for X. Then, there exists a $q \in H_\alpha$ such that $q \Vdash$ '$\dot{X} \notin \dot{I}_\alpha$'. Then, we have $\langle q, \dot{X}\rangle \in H_\alpha * U_\alpha$. So, $\pi_\alpha(\langle q, \dot{X}\rangle) \in \hat{G} * \hat{H}_\alpha$. By the definition of π_α, it follows that $\kappa \in j_\alpha(X)$.

Let $U_0 = U_\alpha \cap V[G]$. By a standard argument, we can show that $(V[G])^\kappa/U_0 \simeq M[\hat{G}]$ and j_0 coincide with the generic elementary embedding induced by U_0.

Define a function $\sigma_\alpha : (V[G][H_\alpha])^\kappa/U_\alpha \to M[\hat{G}][\hat{H}_\alpha]$ as follows. Let $f : \kappa \to V[G][H_\alpha]$ be a function lying in $V[G][H_\alpha]$. Let $\sigma_\alpha([f]_{U_\alpha}) = j_\alpha(f)(\kappa)$. First, we shall show that σ_α is well defined. Suppose that $[f]_{U_\alpha} = [g]_{U_\alpha}$. Let $X = \{\xi < \kappa : f(\xi) = g(\xi)\}$. Then, since $X \in U_\alpha$, we have $\kappa \in j_\alpha(X)$. So, $j_\alpha(f)(\kappa) = j_\alpha(g)(\kappa)$ and hence $\sigma_\alpha([f]_{U_\alpha}) = \sigma_\alpha([g]_{U_\alpha})$. By a similar argument, we can show that $[f]_{U_\alpha} \epsilon_{U_\alpha} [g]_{U_\alpha}$ if and only if $\sigma_\alpha([f]_{U_\alpha}) \in \sigma_\alpha([g]_{U_\alpha})$.

To see that σ_α is onto, let $x \in M[\hat{G}][\hat{H}_\alpha]$. Then, there exists a $j(Q_\alpha)$-name $\dot{x} \in M[\hat{G}]$ such that $x = \dot{x}^{\hat{H}_\alpha}$. Since $(V[G])^\kappa/U_0 \simeq M[\hat{G}]$, there exists a function $f : \kappa \to V[G]$ lying in $V[G]$ such that for every $\xi < \omega_1$, $f(\xi)$ is a Q_α-name and $j_0(f)(\kappa) = \dot{x}$. Define a function $g : \kappa \to V[G][H_\alpha]$ by $g(\xi) = f(\xi)^{H_\alpha}$. Then, $\sigma_\alpha([g]_{U_\alpha}) = j_\alpha(g)(\kappa) = j_0(f)(\kappa)^{\hat{H}_\alpha} = \dot{x}^{\hat{H}_\alpha} = x$.

Therefore, $\sigma_\alpha : (V[G][H_\alpha])^\kappa/U_\alpha \to M[\hat{G}][\hat{H}_\alpha]$ is an isomorphism. Since $M[\hat{G}][\hat{H}_\alpha]$ is well-founded, so is $(V[G][H_\alpha])^\kappa/U_\alpha$. The proof also shows that the generic elementary embedding induced by U_α coincides with j_α. □

Let $H \subseteq Q$ be generic over $V[G]$. Work in $V[G][H]$. It is easy to see that $I = \mathrm{T\breve{C}G}(\overrightarrow{C})$. We shall show that I is precipitous. Suppose not. Then for some generic filter $\tilde{U} \subseteq \mathcal{P}(\omega_1)/I$ over $V[G][H]$, there exists a sequence $\langle f_n : n < \omega\rangle$ in $V[G][H][\tilde{U}]$ such that for every $n < \omega$, f_n is a function from κ into ordinals lying in $V[G][H]$ and $\{\xi < \omega_1 : f_{n+1}(\xi) < f_n(\xi)\} \in \tilde{U}$.

For each $\alpha < \kappa^+$, let $H_\alpha = H \cap Q_\alpha$ and $U_\alpha = \tilde{U} \cap V[G][H_\alpha]$. Since $Q_\alpha * (\mathcal{P}(\omega_1)/\dot{I}_\alpha)$ is regularly embedded into $Q * (\mathcal{P}(\omega_1)/\dot{I})$ by the identity mapping, $H_\alpha * U_\alpha$ is a generic filter of $Q_\alpha * (\mathcal{P}(\kappa)/\dot{I}_\alpha)$ over $V[G]$. By Lemma 3.10, in $V[G][H_\alpha]$, I_α is precipitous. Moreover, if we let $\hat{G} * \hat{H}_\alpha = \pi_\alpha''(H_\alpha * U_\alpha)$, then $(V[G][H_\alpha])^\kappa/U_\alpha \simeq M[\hat{G}][\hat{H}_\alpha]$, and the induced elementary embedding is equal to j_α.

By a standard argument, for every $n < \omega$, there exists an $\alpha_n < \kappa^+$ such that $f_n \in V[G][H_{\alpha_n}]$. Let $\gamma_n = j_{\alpha_n}(f_n)(\kappa)$. We claim that $\gamma_{n+1} < \gamma_n$, which is a contradiction. Since $\{\xi < \omega_1 : f_{n+1}(\xi) < f_n(\xi)\} \in \tilde{U} \cap V[G][H_{\alpha_{n+1}}] = U_{\alpha_{n+1}}$, we have $\gamma_{n+1} = j_{\alpha_{n+1}}(f_{n+1})(\kappa) < j_{\alpha_{n+1}}(f_n)(\kappa)$. But since $j_{\alpha_n} = j_{\alpha_{n+1}} \restriction V[G][H_{\alpha_n}]$, we have $\gamma_n = j_{\alpha_{n+1}}(f_n)(\kappa)$. Therefore, $\gamma_{n+1} < \gamma_n$. Thus, we have proved the following theorem.

Theorem 3.2. *Let* κ *be a measurable cardinal and* ε *an indecomposable ordinal. Then, there exists a forcing extension in which* $\kappa = \aleph_1$ *and there exists a tail club guessing sequence* $\overrightarrow{C}$ *on* ω_1 *of order type* ε *such that* $\mathrm{TCG}(\overrightarrow{C})$ *is precipitous.*

From now on, we assume $V = L[U]$ in addition. We shall show that any restriction of NS_{ω_1} or $\mathrm{TCG}(\overrightarrow{C}')$ for any tail club guessing sequence $\overrightarrow{C}'$ of order type $<\varepsilon$ is not precipitous.

Work in M. For every $\alpha < \kappa^+$, define $\bar{m}_\alpha \in j(P * Q_\alpha)$ to be the truth value that $j^{-1}(\dot{\hat{G}} * \dot{\hat{H}}_\alpha)$ is a generic filter of $P * Q_\alpha$ over M. As in Lemma 3.5, we can show that for every $\beta < \alpha < \kappa^+$, $\bar{m}_\beta = \bar{m}_\alpha \cap j(P * Q_\beta)$.

For every $\alpha < \kappa^+$, define a function $\bar{\pi}_\alpha$ as follows. The domain of $\bar{\pi}_\alpha$ is the set of all triples $\langle p, \dot{q}, \dot{S}\rangle$ such that $p \in P$, $\dot{q}$ is a P-name for an element of $\dot{Q}_\alpha$, and $\dot{S}$ is a $P * \dot{Q}_\alpha$-name for a subset of κ. Let $\langle p, \dot{q}, \dot{S}\rangle$ be in the domain of $\bar{\pi}_\alpha$. Define $\bar{\pi}_\alpha(p, \dot{q}, \dot{S}) = j(\langle p, \dot{q}\rangle) \wedge [\![\kappa \in j(\dot{S})]\!] \in \mathcal{B}(j(P * \dot{Q}_\alpha))$. Note that $j(\dot{S})$ is a $j(P * \dot{Q}_\alpha)$-name for a subset of $j(\kappa)$, so this is a valid definition.

For every $\alpha < \kappa^+$, define a $P * \dot{Q}_\alpha$-name $\dot{\bar{I}}_\alpha$ for a subset of $\mathcal{P}(\kappa)$ so that for every $P * \dot{Q}_\alpha$-name for a subset of κ, $\langle p, \dot{q}\rangle \Vdash$ '$\dot{S} \in \dot{\bar{I}}_\alpha$' if and only if $\bar{\pi}_\alpha(p, \dot{q}, \dot{S})$ is incompatible with $\bar{m}_\alpha$ in $j(P * \dot{Q}_\alpha)$. It is easy to verify that this indeed defines a $P * \dot{Q}_\alpha$-name.

We can easily check that for every $\beta < \alpha < \kappa^+$, if $G * H_\alpha \subseteq P * \dot{Q}_\alpha$ is generic over V and $H_\beta = H_\alpha \cap Q_\beta$, then $\bar{I}_\beta = \mathcal{P}(\kappa)^{V[G][H_\beta]} \cap \bar{I}_\alpha$. Let $\dot{\bar{I}}$ be a $P * \dot{Q}$-name so that $\mathbf{1}_{P * \dot{Q}} \Vdash$ '$\dot{\bar{I}} = \bigcup_{\alpha < \kappa^+} \dot{\bar{I}}_\alpha$'.

Lemma 3.11. *Let $G * H \subseteq P * \dot{Q}$ be generic over V. Let J be a precipitous ideal on κ in $V[G][H]$. Then, $\bar{I} \subseteq J$.*

Proof. We shall show that for every $S \in J^+$, we have $S \in \bar{I}^+$. In $V[G][H]$, let $U_J \subseteq \mathcal{P}(\kappa)/J$ be generic over $V[G][H]$ with $S \in U_J$ and $j_J : V[G][H] \to N_J$ the induced generic elementary embedding. By Lemma 2.13, we have $j_J \restriction V = j$. Define $\hat{G} = j_J(G)$ and $\hat{H} = j_J(H)$. Then, $\hat{G}$ is a generic filter of $j(P)$ over M and $\hat{H}$ is a generic filter of $j_J(Q)$ over $M[\hat{G}]$. For every $x \in V[G][H]$, we have $j_J(x) = j(\dot{x})^{\hat{G} * \hat{H}}$ where $\dot{x}$ is a $P * Q$-name for x.

Let $\dot{S}$ be $P * \dot{Q}$-name for S. There exists an $\alpha < \kappa^+$ such that $\dot{S}$ is a $P * \dot{Q}_\alpha$-name and for some $\langle p, \dot{q}\rangle \in G * (H \cap Q_\alpha)$, $\langle p, \dot{q}\rangle \Vdash$ '$\dot{S} \notin \dot{J}$'. Let $H_\alpha = H \cap Q_\alpha$ and $\hat{H}_\alpha = j_J(H_\alpha)$. Note that $\kappa \in j_J(S)$. Since $j_J(S) = j(\dot{S})^{\hat{G} * \hat{H}_\alpha}$, there exists a $\langle p', \dot{q}'\rangle \in \hat{G} * \hat{H}_\alpha$ such that $\langle p', \dot{q}'\rangle \Vdash$ '$\kappa \in j(\dot{S})$'. So, $\bar{\pi}_\alpha(p, \dot{q}, \dot{S}) \in \hat{G} * \hat{H}_\alpha$. Notice that ${j_J}^{-1}(\hat{G} * \hat{H}_\alpha) = G * H_\alpha$ is generic over V. Therefore, we have $\bar{m}_\alpha \in \hat{G} * \hat{H}_\alpha$. Hence, $\bar{\pi}_\alpha(p, \dot{q}, \dot{S})$ and $\bar{m}_\alpha$ are compatible. It means that $\langle p, \dot{q}\rangle \nVdash$ '$\dot{S} \in \bar{I}_\alpha$'. Since this holds for every sufficiently large $\alpha < \kappa^+$ and every $\langle p, \dot{q}\rangle$ with $\langle p, \dot{q}\rangle \Vdash$ '$\dot{S} \notin \dot{J}$', we have $S \notin \bar{I}$. □

Lemma 3.12. *Suppose $G * H \subseteq P * \dot{Q}$ is generic. For every stationary subset S of ω_1, $\mathrm{NS}_{\omega_1} \restriction S$ is not precipitous in $V[G][H]$.*

Proof. We shall show that for every stationary subset S of ω_1, there exists a stationary subset S' of S such that $S' \in \bar{I}$. It suffices by Lemma 3.11.

Suppose that $\alpha < \kappa^+$, $\langle p, \dot{q}\rangle \in P * \dot{Q}_\alpha$, $\dot{S}$ is a $P * \dot{Q}_\alpha$-name such that $\langle p, \dot{q}\rangle \Vdash$ '$\dot{S}$ is stationary'. Without loss of generality, we may assume that $\langle p, \dot{q}\rangle \Vdash$ '$\dot{X}_\alpha \in \dot{I}_\alpha$'. If $\langle p, \dot{q}\rangle \Vdash$ '$\dot{S} \in \dot{\bar{I}}_\alpha$', then there is nothing to prove. Suppose not. It means that $\bar{\pi}_\alpha(p, \dot{q}, \dot{S})$ is compatible with $\bar{m}_\alpha$. Let $\hat{G} * \hat{H}_\alpha$ be a generic filter of $j(P * \dot{Q}_\alpha)$ over M with $\bar{\pi}_\alpha(p, \dot{q}, \dot{S}), \bar{m}_\alpha \in \hat{G} * \hat{H}_\alpha$. Define $G * H_\alpha = j^{-1}(\hat{G} * \hat{H}_\alpha)$. Since P forces that $|\dot{Q}_\alpha| = \aleph_1$, we have $G * H_\alpha \in M[\hat{G}][\hat{H}_\alpha]$. Since $\bar{m}_\alpha \in \hat{G} * \hat{H}_\alpha$, $G * H_\alpha$ is a generic filter of $P * \dot{Q}_\alpha$ over V. So, we can define $j_\alpha : V[G][H_\alpha] \to M[\hat{G}][\hat{H}_\alpha]$ by letting $j_\alpha(\dot{x}^{G * H_\alpha}) = j(\dot{x})^{\hat{G} * \hat{H}_\alpha}$. Since $\bar{\pi}_\alpha(p, \dot{q}, \dot{S}) \in \hat{G} * \hat{H}_\alpha$, we have $\kappa \in j_\alpha(S)$.

Since $\langle p, \dot{q}\rangle \Vdash$ '$\dot{X}_\alpha \in \dot{I}_\alpha$' and $\langle p, \dot{q}\rangle \in G * H_\alpha$, we have $X_\alpha \in I_\alpha$ in $V[G][H_\alpha]$. So, in $V[G][H_\alpha]$, R_α is the forcing to shoot a $\mathrm{TCG}(\overrightarrow{C})$-measure one set through $\kappa \setminus X_\alpha$. Thus, $j_\alpha(R_\alpha)$ is the forcing to shoot a $\mathrm{TCG}(\overrightarrow{C})$-measure one set through $j_\alpha(\kappa \setminus X_\alpha)$. Define $r' \in j_\alpha(R_\alpha)$ to be the truth value of '$\dot{G}_{R_\alpha} \cap \kappa$ is not generic over $V[G][H_\alpha]$'. There exists a function $f_{r'} : \kappa \to R_\alpha$ such that $j_\alpha(f_{r'})(\kappa) = r'$. Let $\dot{S}'$ be the R_α-name such that for every $\xi \in S$, $[\![\xi \in \dot{S}']\!] = f_{r'}(\xi)$ and for every $\xi \in \omega_1 \setminus S$, $\mathbf{1}_{R_\alpha} \Vdash$ '$\xi \notin \dot{S}$'.

Claim 3.12.1. $\mathbf{1}_{R_\alpha} \Vdash$ '$\dot{S}'$ is stationary'.

Proof. Let $r \in R_\alpha$ and $\dot{D}$ be an R_α-name for a club subset of ω_1. Note $j_\alpha(r) = r$. Let $\langle N_\gamma : \gamma < \omega_1\rangle \in M[G][H_\alpha]$ be a tower of countable elementary submodel of $H(\theta)^{M[G][H_\alpha]}$ for some sufficiently large regular cardinal θ with $r, \dot{D} \in N_0$. For every $\gamma < \omega_1$, let $\delta_\gamma = N_\gamma \cap \omega_1$. In $M[\hat{G}]$, pick an increasing cofinal sequence $\langle \kappa_n : n < \omega\rangle$ in κ. We shall define a decreasing sequence $\langle r_n : n < \omega\rangle$ in R_α and an increasing sequence $\langle \mu_n : n < \omega\rangle$ in κ so that $r_n \in N_{\mu_n+1}$. Let $r_0 = r$ and $\mu_0 = \kappa_0$. Suppose that we have defined r_n so that $r_n \in N_{\mu_n+1}$. Let $\mu_{n+1} < \kappa$ be so large that $\mu_{n+1} \geq \kappa_{n+1}$ and $[\delta_{\mu_n+1}, \delta_{\mu_{n+1}}) \cap j_\alpha(C)_\kappa \neq \varnothing$. Let $r'_n = r_n \cup \{\delta_{\mu_{n+1}} + 1\}$. Then we have $r'_n \in N_{\mu_{n+1}+1} \cap R_\alpha$. Hence, there exists an $r_{n+1} \leq r'_n$ such that $r_{n+1} \in N_{\mu_{n+1}+1}$ and $r_{n+1} \Vdash$ '$\dot{D} \cap [\delta_{\mu_{n+1}} + 1, \delta_{\mu_{n+1}+1}) \neq \varnothing$'.

Define $\tilde{r} = \bigcup_{n<\omega} r_n \cup \{\kappa\}$. It is easy to see that $j_\alpha(C)_\kappa \not\subseteq^* \tilde{r}$. So, we have $\tilde{r} \in j(R_\alpha)$. Since $\tilde{r} \cap \{\delta_\gamma : \gamma < \omega_1\} = \varnothing$, $\tilde{r} \cap \kappa$ cannot be generic over $M[G][H_\alpha]$. It follows that $\tilde{r} \leq r'$. Since $[\![\kappa \in j(\dot{S}')]\!] = r'$, we have $\tilde{r} \Vdash$ '$\kappa \in j(\dot{S}')$'. Since $j_\alpha(\dot{D})$ is a $j_\alpha(R_\alpha)$-name for a club subset and $\tilde{r} \Vdash$ '$\dot{D}$ is unbounded in κ', we have $\tilde{r} \Vdash$ '$\kappa \in j(\dot{D})$'. Therefore, $\tilde{r} \Vdash$ '$j(\dot{S}') \cap j(\dot{D}) \neq \varnothing$'. By elementarity, in $V[G][H_\alpha]$, there exists a $\tilde{r} \leq r$ such that $\tilde{r} \Vdash$ '$\dot{S}' \cap \dot{D} \neq \varnothing$'. Hence $\mathbf{1}_{R_\alpha} \Vdash$ '$\dot{S}'$ is stationary'. □

However, it is easy to see $\mathbf{1}_{R_\alpha} \Vdash$ '$\dot{S}' \in \dot{\bar{I}}_\alpha$'. If we identify $\dot{S}'$ with a $P * \dot{Q}_{\alpha+1}$-name, then we have $\langle p, \dot{q}\rangle \Vdash$ '$\dot{S}' \in \dot{\bar{I}}_\alpha$'. □

The same argument yields the following lemma. Note that Q is $<\varepsilon$-proper, so it is possible to build a decreasing sequence $\langle r_n : n < \omega\rangle$ so that $\tilde{r} \Vdash$ '$j(\dot{C}')_\kappa \subseteq^* j(\dot{D}$').

Lemma 3.13. *Let $G * H \subseteq P * \dot{Q}$ be generic over V. For every indecomposable ordinal $\varepsilon' < \varepsilon$, there is no tail club guessing sequence $\overrightarrow{C}'$ of order type ε' such that $\mathrm{TCG}(\overrightarrow{C})$ is precipitous in $V[G][H]$.*

This finishes the proof of Theorem 3.1.

4. Open problems

While Theorem 3.1 answers some questions asked in [5], it leaves the following question open.

Question 1. Is it consistent that NS_{ω_1} is precipitous but there is a tail club guessing sequence $\overrightarrow{C}$ on ω_1 such that $\mathrm{TCG}(\overrightarrow{C})$ is not precipitous?

In [7], Jech, Magidor, Mitchell, and Prikry constructed from a measurable cardinal that NS_{ω_1} is precipitous. We can show that there is a tail club guessing sequence in this model. If the tail club guessing ideal associated with it is not precipitous, then the previous questions is solved negatively. Otherwise, it solves the following question, which is also interesting.

Question 2. What is the consistency strength that NS_{ω_1} is precipitous and there is a precipitous tail club guessing ideal on ω_1?

Note that the author constructed from a Woodin cardinal a model in which NS_{ω_1} and all tail club guessing ideals on ω_1 are precipitous. Thus, the existence of one Woodin cardinal is an upper bound of the consistency strength. However, it is expected to be much lower.

The following question is not solved in this paper either.

Question 3. Is it consistent that there are two tail club guessing sequences $\overrightarrow{C}$ and $\overrightarrow{C}'$ such that $\mathrm{TCG}(\overrightarrow{C})$ is precipitous, $\mathrm{TCG}(\overrightarrow{C}')$ is not precipitous, and $\overrightarrow{C}$ has smaller order type than $\overrightarrow{C}'$?

The technique used in the proof of Theorem 3.1 seems to be applicable to many other "natural ideals", particularly those associated with guessing principles.

Question 4. Which natural ideals can be precipitous? Are there any relationships between the precipitousness of natural ideals?

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