Minimal Generators of cut-ideals of Graphs without K_4-minors.

Guantao Chen

Department of Mathematics and Statistics
Georgia State University
Atlanta, Georgia

Co-author: Joseph Brennan

March 16, 2007

Basic Notation

- $\mathbb{K}$: a field;
- $G = (V, E)$: a graph;
- $A \cup B$: a partition of V;
- $A|B := E[A, B]$: the set of edges ab such that $a \in A$ and $b \in B$;
- Coordinates $q_{A|B}$: associated with cuts $A|B$, clearly, $q_{A|B} = q_{B|A}$.
- Coordinates $s_e,\ t_e$: associated with each edge $e \in E$.

Polynomials Rings and Cut Ideals

Definition

We consider two polynomial rings on these two sets of unknowns.

$$\begin{aligned} \mathbb{K}[q] &:= \mathbb{K}[q_{A|B} \mid A|B \text{ is a cut of } G], \\ \mathbb{K}[s,t] &:= \mathbb{K}[s_e, t_e \mid e \in E]. \end{aligned}$$

Polynomials Rings and Cut Ideals

Definition

We consider two polynomial rings on these two sets of unknowns.

$$\begin{aligned} \mathbb{K}[q] &:= \mathbb{K}[q_{A|B} \mid A|B \text{ is a cut of } G], \\ \mathbb{K}[s,t] &:= \mathbb{K}[s_e, t_e \mid e \in E]. \end{aligned}$$

Definition

Let ϕ be a homomorphism of polynomial rings:

$$\phi_G \ : \ \mathbb{K}[q] \rightarrow \mathbb{K}[s,t], \quad q_{A|B} \mapsto \Pi_{e \in A|B} s_e \Pi_{e \notin A|B} t_e. \tag{1}$$

Polynomials Rings and Cut Ideals

Definition

We consider two polynomial rings on these two sets of unknowns.

$$\begin{aligned} \mathbb{K}[q] &:= \mathbb{K}[q_{A|B} \mid A|B \text{ is a cut of } G], \\ \mathbb{K}[s,t] &:= \mathbb{K}[s_e, t_e \mid e \in E]. \end{aligned}$$

Definition

Let ϕ be a homomorphism of polynomial rings:

$$\phi_G \ : \ \mathbb{K}[q] \to \mathbb{K}[s,t], \quad q_{A|B} \mapsto \Pi_{e \in A|B} s_e \Pi_{e \notin A|B} t_e. \qquad (1)$$

Observation

The Kernel I_G of ϕ, cut ideal of G, is a homogeneous toric ideal generated by binomials.

An Example

Example

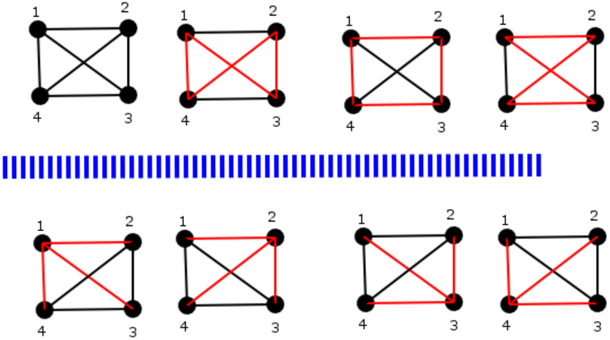
$G = K_4$ with $V = \{1,2,3,4\}$ and $E = \{12,13,14,23,24,34\}$. ϕ_{K_4} is specified by

$$\begin{array}{ll} q_{|1234} \mapsto t_{12}t_{13}t_{14}t_{23}t_{24}t_{34} & q_{1|234} \mapsto s_{12}s_{13}s_{14}t_{23}t_{24}t_{34} \\ q_{12|34} \mapsto t_{12}s_{13}s_{14}s_{23}s_{24}t_{34} & q_{2|134} \mapsto s_{12}t_{13}t_{14}s_{23}s_{24}t_{34} \\ q_{13|24} \mapsto s_{12}t_{13}s_{14}s_{23}t_{24}s_{34} & q_{3|124} \mapsto t_{12}s_{13}t_{14}s_{23}t_{24}s_{34} \\ q_{14|23} \mapsto t_{12}s_{13}t_{14}s_{23}s_{24}s_{34} & q_{4|123} \mapsto t_{12}t_{13}s_{14}t_{23}s_{24}s_{34} \end{array}$$

and

$$I_{K_4} = < q_{|1234}q_{12|34}q_{13|24}q_{14|23} - q_{1|234}q_{2|134}q_{3|124}q_{4|123} > .$$

K_4

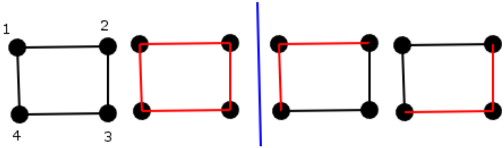


An Example

Example

$G = C_4$ with $V = \{1,2,3,4\}$ and $E = \{12, 23, 34, 14\}$. The ring map ϕ_{C_4} is derived from ϕ_{K_4} in the previous example by setting $s_{13} = t_{13} = s_{24} = t_{24} = 1$.

$$I_{C_4} = \quad < q_{|1234}q_{13|24} - q_{1|234}q_{3|124}, \\ q_{|1234}q_{13|24} - q_{2|134}q_{4|123}, q_{|1234}q_{13|24} - q_{12|34}q_{14|23} > .$$



Cut-Space

We note that the exponent vector of $\phi(A|B)$ of s variables is the cut-vector in the edge-cut space.

Known Results

Definition

Let $\mu(G)$ be the largest degree of a minimal generator of I_G.

Known Results

Definition
Let $\mu(G)$ be the largest degree of a minimal generator of I_G.

Example (Sturmfels and Sullivant)
$\mu(G)$ for all graphs G of order $n \leq 6$ are calculated. In particular,

Known Results

Definition
Let $\mu(G)$ be the largest degree of a minimal generator of I_G.

Example (Sturmfels and Sullivant)
$\mu(G)$ for all graphs G of order $n \leq 6$ are calculated. In particular,

- $\mu(K_1) = \mu(K_2) = \mu(K_3) = 0$;

Known Results

Definition

Let $\mu(G)$ be the largest degree of a minimal generator of I_G.

Example (Sturmfels and Sullivant)

$\mu(G)$ *for all graphs* G *of order* $n \leq 6$ *are calculated. In particular,*

- $\mu(K_1) = \mu(K_2) = \mu(K_3) = 0$;
- $\mu(C_4) = \mu(C_5) = \mu(C_6) = 2$;

Known Results

Definition

Let $\mu(G)$ be the largest degree of a minimal generator of I_G.

Example (Sturmfels and Sullivant)

$\mu(G)$ for all graphs G of order $n \leq 6$ are calculated. In particular,

- $\mu(K_1) = \mu(K_2) = \mu(K_3) = 0$;
- $\mu(C_4) = \mu(C_5) = \mu(C_6) = 2$;
- $\mu(K_4) = 4$, $\mu(K_5) = 6$, $\mu(K_6) = 10$;

Known Results

Definition

Let $\mu(G)$ be the largest degree of a minimal generator of I_G.

Example (Sturmfels and Sullivant)

$\mu(G)$ for all graphs G of order $n \leq 6$ are calculated. In particular,

- $\mu(K_1) = \mu(K_2) = \mu(K_3) = 0$;
- $\mu(C_4) = \mu(C_5) = \mu(C_6) = 2$;
- $\mu(K_4) = 4$, $\mu(K_5) = 6$, $\mu(K_6) = 10$;
- $\mu(K_{2,3}) = 2$, $\mu(K_{3,3}) = 4$, $\mu(K_{2,2,2}) = 4$;

Known Results

Definition

Let $\mu(G)$ be the largest degree of a minimal generator of I_G.

Example (Sturmfels and Sullivant)

$\mu(G)$ for all graphs G of order $n \leq 6$ are calculated. In particular,

- $\mu(K_1) = \mu(K_2) = \mu(K_3) = 0$;
- $\mu(C_4) = \mu(C_5) = \mu(C_6) = 2$;
- $\mu(K_4) = 4$, $\mu(K_5) = 6$, $\mu(K_6) = 10$;
- $\mu(K_{2,3}) = 2$, $\mu(K_{3,3}) = 4$, $\mu(K_{2,2,2}) = 4$;
- $\mu(K_2 \times K_3) = 4$.

Graphs with Small Intersection

Theorem (Sturmfels and Sullivant)
If $G = G_1 \cup G_2$ *and* $G_1 \cap G_2 = K_1$ *or* K_2, *then* $\mu(G) \leq \max\{\mu(G_1), \mu(G_2)\}$. *Consequently,* I_T *is generated by quadrics provided* T *is a tree.*

Corollary
Let T *be a tree. Then,* $\mu(T) \leq 2$.

Conjectures of Sturmfels and Sullivant

Conjecture

Let H be obtained from G by deleting an edge. Then $\mu(I_H) \leq \mu(I_G)$.

Conjecture

$\mu(G) \leq 2$ *if and only if G does not contain* K_4 *as a minor.*

Conjecture

$\mu(G) \leq 4$ *if and only if G does not contain* K_5 *as a minor.*

Switching Lemma

Lemma

Let C be a cycle and $A|B$ be a cut of C. Then the following statements hold:

1. *For any $e_1, e_2 \notin A|B$ there exists a cut $A'|B'$ such that $A'|B' = A|B \cup \{e_1, e_2\}$;*

Switching Lemma

Lemma

Let C be a cycle and $A|B$ be a cut of C. Then the following statements hold:

1. *For any $e_1,\ e_2 \notin A|B$ there exists a cut $A'|B'$ such that* $A'|B' = A|B \cup \{e_1, e_2\}$;
2. *For any $e_1 \in A|B$ and $e_2 \notin A|B$ there exists a cut $A'|B'$ such that* $A'|B' = (A|B - e_1) \cup \{e_2\}$;

Switching Lemma

Lemma

Let C be a cycle and $A|B$ be a cut of C. Then the following statements hold:

1. *For any $e_1,\ e_2 \notin A|B$ there exists a cut $A'|B'$ such that* $A'|B' = A|B \cup \{e_1, e_2\}$;
2. *For any $e_1 \in A|B$ and $e_2 \notin A|B$ there exists a cut $A'|B'$ such that* $A'|B' = (A|B - e_1) \cup \{e_2\}$;
3. *For any $e_1, e_2 \notin A|B$ there exists a cut $A'|B'$ such that* $A'|B' = A|B - \{e_1, e_2\}$.

Differences of Cuts

Lemma

Let G be a 2-connected graph and $A|B$ and $C|D$ be two cuts of G. If $A|B \neq C|D$, then $\delta(A|B, C|D) \geq 2$, where $\delta(X, Y) = |X - Y| + |Y - X|$ is the cardinality of the symmetric difference of X and Y.

frametitleEdge Switching Candidates

Lemma

Let G be a graph and let $\prod q_{A_i|B_i} - \prod_{C_i|D_i} \in I_G$. If there exists two edges $e_1,\ e_2 \in A_1|B_1 - C_1|D_1$, then either there exists $A_j|B_j$ such that $e_1, e_2 \notin A_j|B_j$ or there exists $C_j|D_j$ such that $e_1, e_2 \in C_j|D_j$.

frametitleEdge Switching Candidates

Lemma

Let G be a graph and let $\Pi q_{A_i|B_i} - \Pi_{C_i|D_i} \in I_G$*. If there exists two edges* $e_1,\ e_2 \in A_1|B_1 - C_1|D_1$*, then either there exists* $A_j|B_j$ *such that* $e_1, e_2 \notin A_j|B_j$ *or there exists* $C_j|D_j$ *such that* $e_1, e_2 \in C_j|D_j$*.*

Lemma

Let G be a graph and let $\Pi q_{A_i|B_i} - \Pi_{C_i|D_i} \in I_G$*. If there exists two edges* e_1 *and* e_2 *such that* $e_1 \in A_1|B_1 - C_1|D_1$ *and* $e_2 \in C_1|D_1 - A_1|B_1$*, then either there exists* $A_j|B_j$ *such that* $e_1 \notin A_j|B_j$ *and* $e_2 \in A_j|B_j$ *or there exists* $C_j|D_j$ *such that* $e_1 \in C_j|D_j$ *and* $e_2 \notin C_j|D_j$*.*

cycles

Theorem

$\mu(C) \leq 2$ *for every cycle* C.

cycles

Theorem

$\mu(C) \leq 2$ *for every cycle* C*.*

- We only need to show that every binormial $\Pi_{i=1}^{m} q_{A_i|B_i} - \Pi_{i=1}^{m} q_{C_i|B_i} \in I_C$ can be generated by quadrics.

cycles

Theorem

$\mu(C) \leq 2$ *for every cycle* C*.*

- We only need to show that every binormial $\Pi_{i=1}^{m} q_{A_i|B_i} - \Pi_{i=1}^{m} q_{C_i|B_i} \in I_C$ can be generated by quadrics.
- We induction on m based the following result.

Common Edges

Lemma

Let $\prod_{i=1}^{m} q_{A_i|B_i} - \prod_{i=1}^{m} q_{C_i|B_i} \in I_C$ *and assume that* $A_1|B_1 \neq C_1|D_1$*. Then there exist cuts* $A_i'|B_i'$ *and* $C_i'|D_i'$ *for* $i = 1, 2, \ldots, m$ *such that the following statements hold.*

1. $\delta(A_1'|B_1', C_1'|D_1') < \delta(A_1|B_1, C_1|D_1)$;

Common Edges

Lemma

Let $\prod_{i=1}^{m} q_{A_i|B_i} - \prod_{i=1}^{m} q_{C_i|B_i} \in I_C$ *and assume that* $A_1|B_1 \neq C_1|D_1$*. Then there exist cuts* $A'_i|B'_i$ *and* $C'_i|D'_i$ *for* $i = 1, 2, \ldots, m$ *such that the following statements hold.*

1. $\delta(A'_1|B'_1, C'_1|D'_1) < \delta(A_1|B_1, C_1|D_1)$;
2. *there exists a* j *such that* $q_{A_1|B_1} q_{A_j|B_j} - q_{A'_1|B'_1} q_{A'_j|B'_j} \in I_C$ *and* $q_{C_1|D_1} q_{C_j|D_j} - q_{C'_1|D'_1} q_{C'_j|D'_j} \in I_C$.

Proof -1

- Using Lemma 14, we have $\delta(A_1|B_1, C_1|D_1) \geq 2$.

Proof -1

- Using Lemma 14, we have $\delta(A_1|B_1, C_1|D_1) \geq 2$.
- Let $e_1, e_2 \in \Delta(A_1|B_1, C_1|D_1)$.

Proof -1

- Using Lemma 14, we have $\delta(A_1|B_1, C_1|D_1) \geq 2$.
- Let $e_1, e_2 \in \Delta(A_1|B_1, C_1|D_1)$.
- We will only consider two cases (I) $e_1, e_2 \in A_1|B_1 - C_1|D_1$ and (II) $e_1 \in A_1|B_1 - C_1|D_1$ and $e_2 \in C_1|D_1 - A_1|B_1$ since the others cases can be considered similarly.

Proof -1

- Using Lemma 14, we have $\delta(A_1|B_1, C_1|D_1) \geq 2$.
- Let $e_1, e_2 \in \Delta(A_1|B_1, C_1|D_1)$.
- We will only consider two cases (I) $e_1, e_2 \in A_1|B_1 - C_1|D_1$ and (II) $e_1 \in A_1|B_1 - C_1|D_1$ and $e_2 \in C_1|D_1 - A_1|B_1$ since the others cases can be considered similarly.
- In fact, we will only prove the case (I) in this talk.

(I) $e_1, e_2 \in A_1|B_1 - C_1|D_1$

- Using Lemma 15, WLOG, we assume that $e_1, e_2 \notin A_2|B_2$;

(I) $e_1, e_2 \in A_1|B_1 - C_1|D_1$

- Using Lemma 15, WLOG, we assume that $e_1, e_2 \notin A_2|B_2$;
- We find a cut $A_1'|B_1'$ such that $A_1'|B_1' = A_1|B_1 - \{e_1, e_2\}$;

(I) $e_1, e_2 \in A_1|B_1 - C_1|D_1$

- Using Lemma 15, WLOG, we assume that $e_1, e_2 \notin A_2|B_2$;
- We find a cut $A_1'|B_1'$ such that $A_1'|B_1' = A_1|B_1 - \{e_1, e_2\}$;
- We find a cut $A_2'|B_2'$ such that $A_2'|B_2' = A_2|B_2 \cup \{e_1, e_2\}$.

(I) $e_1, e_2 \in A_1|B_1 - C_1|D_1$

- Using Lemma 15, WLOG, we assume that $e_1, e_2 \notin A_2|B_2$;
- We find a cut $A_1'|B_1'$ such that $A_1'|B_1' = A_1|B_1 - \{e_1, e_2\}$;
- We find a cut $A_2'|B_2'$ such that $A_2'|B_2' = A_2|B_2 \cup \{e_1, e_2\}$.
- $q_{A_1|B_1} q_{A_2|B_2} - q_{A_1'|B_1'} q_{A_2'|B_2'} \in I_C$;

(I) $e_1, e_2 \in A_1|B_1 - C_1|D_1$

- Using Lemma 15, WLOG, we assume that $e_1, e_2 \notin A_2|B_2$;
- We find a cut $A_1'|B_1'$ such that $A_1'|B_1' = A_1|B_1 - \{e_1, e_2\}$;
- We find a cut $A_2'|B_2'$ such that $A_2'|B_2' = A_2|B_2 \cup \{e_1, e_2\}$.
- $q_{A_1|B_1} q_{A_2|B_2} - q_{A_1'|B_1'} q_{A_2'|B_2'} \in I_C$;
- $q_{A_1'|B_1'} q_{A_2'|B_2'} \prod_{i \geq 3} q_{A_i|B_i} - \prod q_{C_i|D_i} \in I_C$.

Outerplanar Graphs

Theorem

If G is an outerplanar graph, then $\mu(G) \leq 2$.

Outerplanar Graphs

Theorem

If G is an outerplanar graph, then $\mu(G) \leq 2$.

- Induction on order of G.

Outerplanar Graphs

Theorem

If G is an outerplanar graph, then $\mu(G) \leq 2$.

- Induction on order of G.
- We may assume that G is connected;

Outerplanar Graphs

Theorem

If G is an outerplanar graph, then $\mu(G) \leq 2$.

- Induction on order of G.
- We may assume that G is connected;
- We may assume that G is 2-connected;

Outerplanar Graphs

Theorem
If G is an outerplanar graph, then $\mu(G) \leq 2$.

- Induction on order of G.
- We may assume that G is connected;
- We may assume that G is 2-connected;
- We may assume that G has a vertex-cut $\{u, v\}$ such that $uv \in E$;

Outerplanar Graphs

Theorem

If G is an outerplanar graph, then $\mu(G) \leq 2$.

- Induction on order of G.
- We may assume that G is connected;
- We may assume that G is 2-connected;
- We may assume that G has a vertex-cut $\{u, v\}$ such that $uv \in E$;
- Then, G is a cycle.