# A proof of the stability of extremal graphs.

Zoltán Füredi

University of Illinois at Urbana-Champaign, and

Hungarian Academy of Sciences, Budapest

March, 2007. Oxford, OH

ABSTRACT:

We present a concise, contemporary proof (i.e., one using Szemerédi's regularity lemma) for the following classical stability result of Simonovits 1968:

If an $n$-vertex $F$-free graph $G$ is almost extremal, $\mathrm{chr}(F) = p+1$, then the structure of $G$ is close to a $p$-partite Turán graph.
More precisely, for every graph $F$ and $\varepsilon > 0$ there exists a $\delta > 0$ and a bound $n_0$ (depending on $F$ and $\varepsilon$) such that if $n > n_0$ and

$$e(G) > (1 - \frac{1}{p})\binom{n}{2} - \delta n^2$$

then one can change (add and delete) at most $\varepsilon n^2$ edges of $G$ and obtain a complete $p$-partite graph.

## AIM OF LECTURE:

proofs for 3 classical Extremal Graph Theorem

1. Turán's problem, asymptotic
2. First proof: Erdős' proof for Turan
3. Stability of $\mathbf{ex}(n, K_{p+1})$
4. Third proof: Appl'n of Regularity lemma

# Some notation

$[n] := \{1, 2, \ldots, n\}$

$G_n$ graph on $n$ vertices

$\chi(G) :=$ chromatic number,

$e(G) :=$ number of edges,

$T_{n,p}$ := Turán graph, the $p$-chromatic graph having the most edges.

$\deg_G(x)$ degree of vertex $x$ of graph $G$,
$N_G(x) \subset V$, neighborhood

and

$N_G(x, A) := A \cap N_G(x)$, neighborhood in $A$,
$\deg_G(x, A)$ degree of vertex $x$ to $A \subset V(G)$.

# Turán's theorem
# Turán type graph problems

THM. Mantel (1903).

Turán(1940)

$$e(G_n) > e(T_{n,p}) \quad \Longrightarrow \quad K_{p+1} \subseteq G_n.$$

General question: Given a family $\mathcal{F}$ of forbidden graphs, what is the maximum of $e(G_n)$ if $G_n$ does not contain subgraphs $F \in \mathcal{F}$?

Notation: $\mathbf{ex}(n,\mathcal{F}) := \max e(G)$

$\mathbf{ex}(n, K_{p+1}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + O(n).$

Unique extremal graph.

# General asymptotics

**Erdős-Stone-Simonovits** (1946), (1966)

If

$$\min_{F\in\mathcal{F}} \chi(F) = p + 1$$

then

$$\mathbf{ex}(n, \mathcal{F}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

The asymptotics depends only on the
**minimum chromatic number**

**Corollary 1** $P_{10} =$ *Petersen graph.*

$$\mathbf{ex}(n, P_{10}) = \frac{1}{4}n^2 + o(n^2).$$

# Structural stability

For every $\varepsilon > 0$ and $\mathcal{F}$ there is a $\delta > 0$, and $n_0$ such that if

$$\min_{F \in \mathcal{F}} \chi(F) = p + 1,$$

$F \not\subseteq G_n$, $n > n_0$ and

$$e(G_n) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - \delta n^2,$$

then

$$E(G_n) \triangle E(T_{n,p}) \leq \varepsilon n^2.$$

I.e., one can change (add and delete) at most $\varepsilon n^2$ edges of $G$ and obtain a complete $p$-partite graph.

AIM of talk: to present a new proof for this Stability Theorem.

# Stability of $\mathbf{ex}(n, K_{p+1})$

**Theorem 1**
*Suppose* $K_{p+1} \not\subset G$, $|V(G)| = n$ *and*

$$e(G) \geq e(T_{n,p}) - t.$$

*Then there exists a* $p$*-chromatic subgraph* $H_0$, $E(H_0) \subset E(G)$ *such that*

$$e(H_0) \geq e(G) - t.$$

**Corollary 2**

*$\exists$ a complete $p$-chromatic graph $H$, such that*

$$|E(G) \triangle E(H)| \le 3t.$$

Proof: Delete $t$ edges to make it $p$-partite,
add at most $2t$ to make it complete $p$-partite.

There are other (more exact) stability results (see below.)
Advantage of this one: No $\varepsilon, \delta, n_0$
it is true for every $n$ and $t$.

**Large $p$-chromatic subgraphs**:

If $t < n/(2p) - O(1)$ then $G$ itself is $p$-chromatic (Hanson, Toft 1991), there is no need to delete.

One can always delete at most $e/2$ edges to make $G$ bipartite (Erdős)
Generalizations, e.g., Lovász 1976, Alon 1996, Tuza et al.

E. Győri 1987, 1991

$$e(H_0) \geq e(G) - O(t^2/n^2).$$

# Degree majorization

**Theorem 2** (Erdős, 1970)
*Suppose* $K_{p+1} \not\subset G$, *then* $\exists$ *a* $p$*-chromatic* $H$ *on the same vertex set*,

$$V(H) = V(G)$$

*majorizing the degrees:*

$$\deg_H(x) \geq \deg_G(x)$$

*for every* $x \in V$.

**Proof** of Erdős' degree majorization: (An Algorithm.)

Input: $G$

Output: $V_1, V_2, \ldots, V_p$ a $p$-partition of $V(G)$.

$H := K(V_1, \ldots, V_p)$ a $p$-partite complete graph

Let $x_1 :=$ a vertex with max degree.

Let $V_1 := V \setminus N(x_1)$.

Let $x_i :=$ a vertex with max degree on the graph of the rest of the vertices, of $G|(V_1 \cup \ldots V_{i-1})$.

Procedure stops in $p$ steps, $\{x_1, x_2, \ldots, x_p\}$ spans a complete graph. Q.E.D.

**Proof** of stability of $K_{p+1}$ ($\exists$ large $p$-partite $H_0 \subset G$)

Algorithm. Input: $G$

Output: $V_1, V_2, \ldots, V_p$, a partition of $V(G)$ such that

$$\sum_i e(G|V_i) \le t.$$

Consider the previous partition $V_1, V_2, \ldots, V_p$, $d_i = \deg(x_i)$. We have $\deg_{G|V_i \cup V_{i+1} \cdots \cup V_p}(x) \le d_i$ for $x_i \in V_i$. Then

$$e(G) \le \sum |V_i| \times d_i = e(K(V_1, V_2, \ldots, V_p)) \le e(T_{n,p}).$$

However! edges inside $V_i$ are counted twice:

$$\sum_i e(G|V_i) \le e(T_{n,p}) - e(G) \le t. \quad \square$$

# Proof of stability for $F$

Tools:

Corollary 2, i.e., stability for $K_{p+1}$.

(We suppose $\mathrm{chr}(F) = p + 1$.)

&

Szemerédi's regularity lemma, more exactly

'Counting lemma' + 'Removal lemma'.

Basic notion of quasi-randomness:

DEF: $G(A, B)$ bipartite graph is $\alpha$-**regular**, ($\alpha$-quasi random) if for all $A' \subset A$, $|A'| \geq \alpha|A|$ and $B' \subset B$, $|B'| \geq \alpha|B|$ one has

$$|d_G(A', B') - d_G(A, B)| \leq \alpha,$$

where

$$d_G(A', B') = e(G[A', B'])/|A'||B'|$$

is the **density** of the induced bipartite subgraph $G[A', B']$.

# The Regularity Lemma

**Theorem 3** (Szemerédi 1978)
*For every* $\alpha > 0$ *and integer* $\ell_0$, *there exist integers* $L_0 = L_0(\alpha, \ell_0)$ *and* $n_0 = n_0(\alpha, \ell_0)$ *so that for* **every** *graph* $G = (V, E)$, $|V| \geq n_0$, *$V$ admits a partition* $V = V_1 \cup \ldots \cup V_L$, $\ell_0 \leq L \leq L_0$, *satisfying*
(i) $|V_1| \leq |V_2| \leq \ldots \leq |V_L| \leq |V_1| + 1$ *and*
(ii) *all but at most* $\alpha\binom{L}{2}$ *pairs* $(V_i, V_j)$, $1 \leq i < j \leq L$, *are* $\alpha$*-regular.*

# The Reduced Graph

$G \to R$ **$\beta$-reduced graph**
where $R$ has $L$ vertices $\{1, 2, \ldots, L\}$ and
$(i, j) \in E(R)$ if $(V_i, V_j)$ is $\alpha$-regular with density $\geq \beta$.

# The Graph Counting Lemma

Counting embeddings, homomorphisms.

Folklore.

Early forms: Szemerédi, Ruzsa & Szemerédi.

Contemporary forms: Lovász et al.

Surveys: Komlós-Simonovits 1996,

Komlós-Shokoufandeh-Simonovits-Szemerédi 2002.

For hypergraphs: Rödl, Nagle, Skokan, Solymosi, Tao, Gowers etc.

A simple form of graph counting lemma:

**Theorem 4** *For all densities* $\beta > 0, \gamma > 0$ *and every positive integer* $\ell$*, there exist* $\alpha > 0$ *and* $n_0$ *so that whenever* $G$ *is an* $\ell$*-partite graph with* $\ell$*-partition* $V_1 \cup \ldots \cup V_\ell$, $|V_i| = n > n_0$*, satisfying for all* $1 \leq i < j \leq \ell$

(i) $d_G(V_i, V_j) = \beta \pm \alpha$ *and*

(ii) $(V_i, V_j)$ *is* $\alpha$*-regular,*

*then the number of* $\ell$*-cliques in* $G$ *is*

$$\beta^{\binom{\ell}{2}} n^\ell (1 \pm \gamma).$$

COROLLARY: If $\ell = p + 1$, then $G$ is NOT $F$-free.

# The Removal Lemma

Early forms: Ruzsa & Szemerédi.

Surveys: Komlós-Simonovits 1996,

Komlós-Shokoufandeh-Simonovits-Szemerédi 2002.

$r$-graphs: Rödl-Nagle-Skokan, Solymosi, Tao, Gowers

**Theorem 5** *For* $\forall\ \gamma > 0$ *and fixed graph* $F$, *chr*$(F) = p + 1$, $\exists\ c > 0$ *and* $n_0$ *so that whenever* $G$ *has at most* $cn^{p+1}$ *copies of* $F$, *then one may remove* $\gamma n^2$ *edges of* $G$ *to obtain a (large) subgraph* $G_0$ *of* $G$ *which is* $F$*-free. I.e.,* $G_0$ *contains no copy of* $F$ *at all.*

# Proof of stability for $F$

We suppose chr$(F) = p + 1$.
Given $G$ with $n > n_0$, $e(G) > (1 - \frac{1}{p})\binom{n}{2} - \delta n^2$.
($\delta$ will be defined later), $G$ is $F$-free.

Apply Szemerédi's regularity lemma to $G$, with $\alpha > 0$.
Obtain regular partition $V_1, \ldots, V_L$.
Leave out edges of
– irregular pairs
– inner edges (inside $V_i$'s)
– low density pairs (less than density $\beta$).

Consider reduced graph $R$ on $\{1, 2, \ldots, L\}$.

By removal/counting lemma: $R$ is $K_{p+1}$-free.
So $e(R) \leq (1 - \frac{1}{p})\binom{L}{2} + L$ by Turán.

On the other hand

$$e(R)(\frac{n}{L})^2 + O(\alpha + \beta)n^2 > e(G) > \frac{p-1}{2p}n^2 - \delta n^2.$$

Hence $e(R) > e(T_{L,p}) - (\alpha + \beta + \delta)L^2$.

Remainder term: $t$.

Use stability for $K_{p+1}$:
One can change $3t$ edges of $R$ to get a complete $p$-partite one.
This corresponds changing another $O(tn^2/L^2)$ edges of $G_0$ to make it complete $p$-partite.
Q.E.D.