# Equitable colorings of graphs with bounded vertex degrees

H. Kierstead and A. Kostochka

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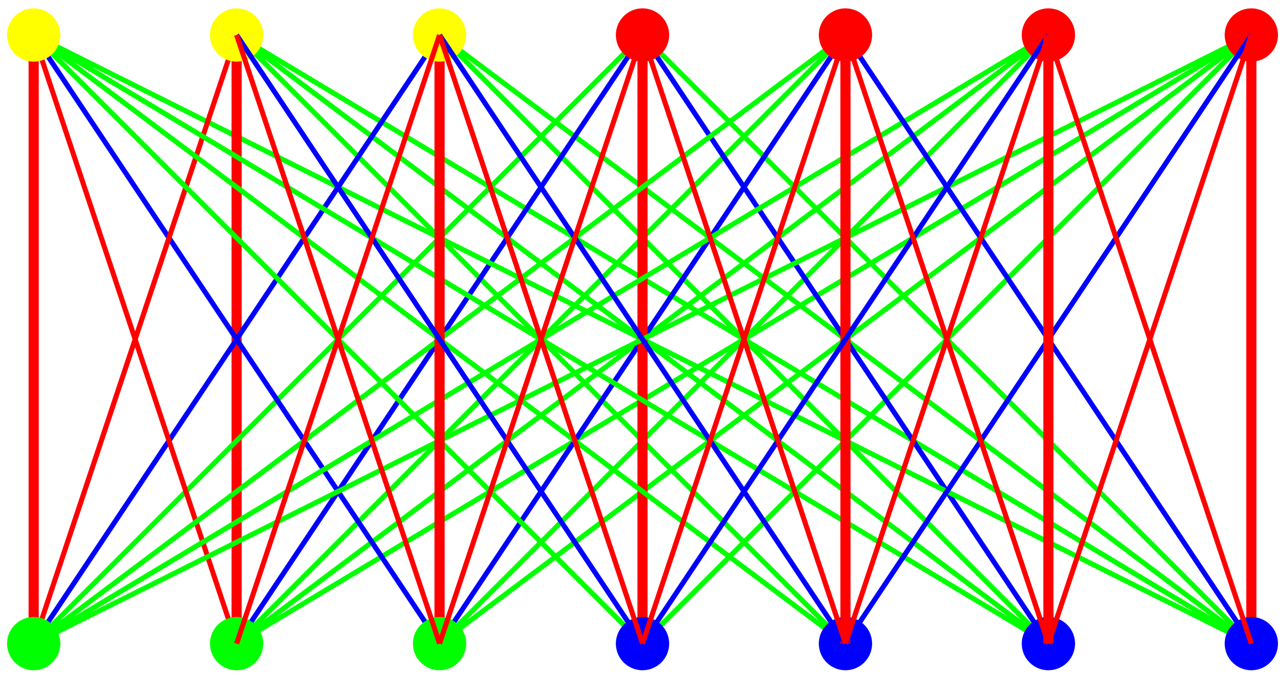
Applications: scheduling, partitioning, and load balancing problems; graph and hypergraph packing problems; deviation bounds for sums of random variables with limited dependence.

A graph may have an equitable $k$-coloring but have no any equitable $(k+1)$-coloring.



Let $eq(G) = \min\{k \; :$
$G$ has an equitable $m$-coloring for each $m \geq k\}$

To decide whether a graph has an equitable $k$-coloring is $NP$-complete even for $k = 3$.

This motivates extremal problems: if a graph $G$ is sparse, then it has low $eq(G)$.

“Sparse” may mean “Low maximum degree”, or “Low average degree”, or “Low degeneracy”, or a combination of those, or something else.

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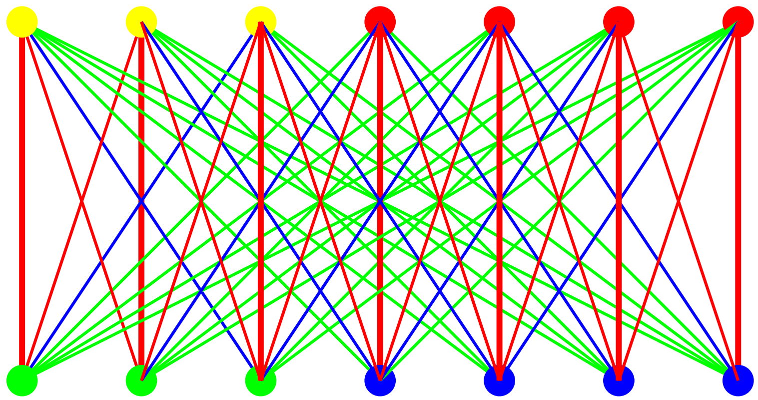

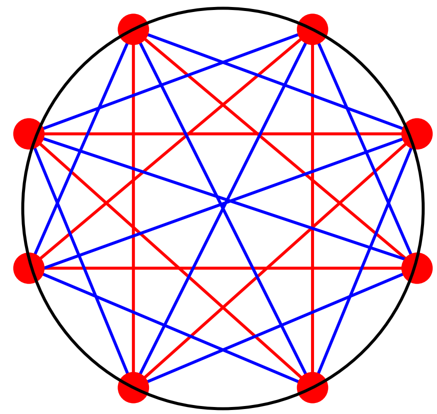

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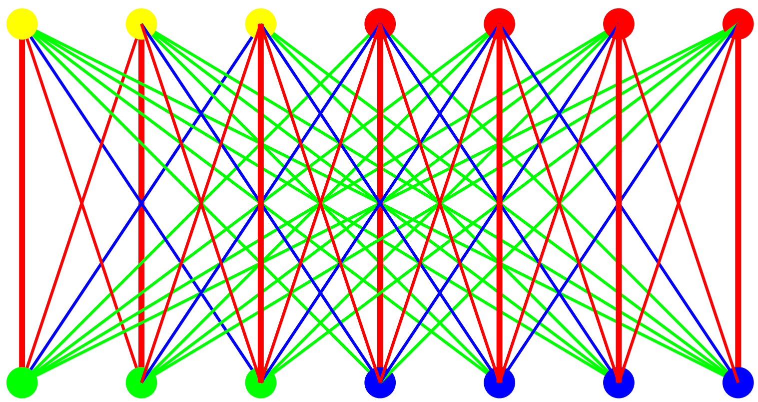

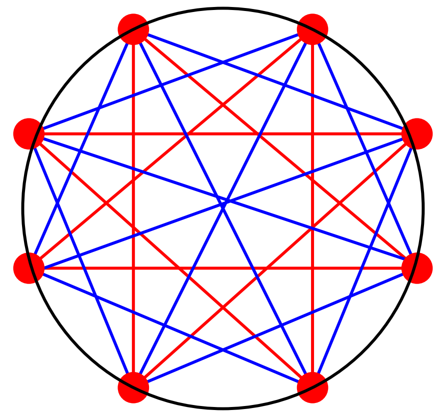

Conjecture 2. [Chen-Lih-Wu] Let $G$ be a connected graph with maximum degree at most $r$. If $G$ is distinct from $K_{r+1}$, $K_{r,r}$ (for odd $r$), and is not an odd cycle, then $G$ has an equitable $r$-coloring.

The Chen-Lih-Wu Conjecture was proved:

1) For $r \leq 3$ [Chen-Lih-Wu]

2) For bipartite graphs [Lih-Wu]

3) For interval graphs [Chen-Lih-Yan]

4) For split graphs [Chen-Ko-Lih]

5) For outerplanar graphs [Yap-Zhang]

6) For planar graphs $G$ with $\Delta(G) \geq 13$ [Yap-Zhang]

7) For planar graphs $G$ with $\Delta(G) \geq 9$ [Nakprasit]

8) For graphs $G$ with $avdeg(G) \leq \Delta(G)/5$ [Kostochka-Nakprasit]

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Let the total edge degree of $G$ be

$$\theta(G)=\max\{d(x)+d(y)\ :\ xy\in E(G)\}.$$

$$\delta(G)+\Delta(G)\leq\theta(G)\leq 2\Delta(G).$$

$$\theta(G)=\Delta(L(G))+2.$$

$\theta(G)$ equals the maximum degree of an edge of $G$ in the total graph $T(G)$.

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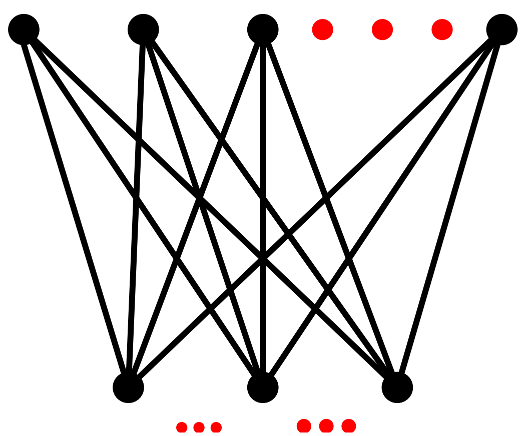

$\mathbf{K_{m,2r-m}}$ **(odd m)**

Conjecture 4. [K-K] If $r \geq 3$ and a connected graph $G$ with $\theta(G) \leq 2r$ differs from $K_{r+1}$ and $K_{m,2r-m}$ for all odd $m$, then $G$ has $eq(G) \leq r$.

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Theorem 11. [K-K] Conjecture 4 holds for $r = 3$.

For odd $r \geq 3$, the Chen-Lih-Wu Conjecture does not describe disconnected graphs with max.deg $r$ that are not equitably $r$-colorable. For example, for an odd $r$, $K_{r,r} \cup K_{r,r}$ is equitably $r$-colorable, but $K_{r,r} \cup K_r$ is not.

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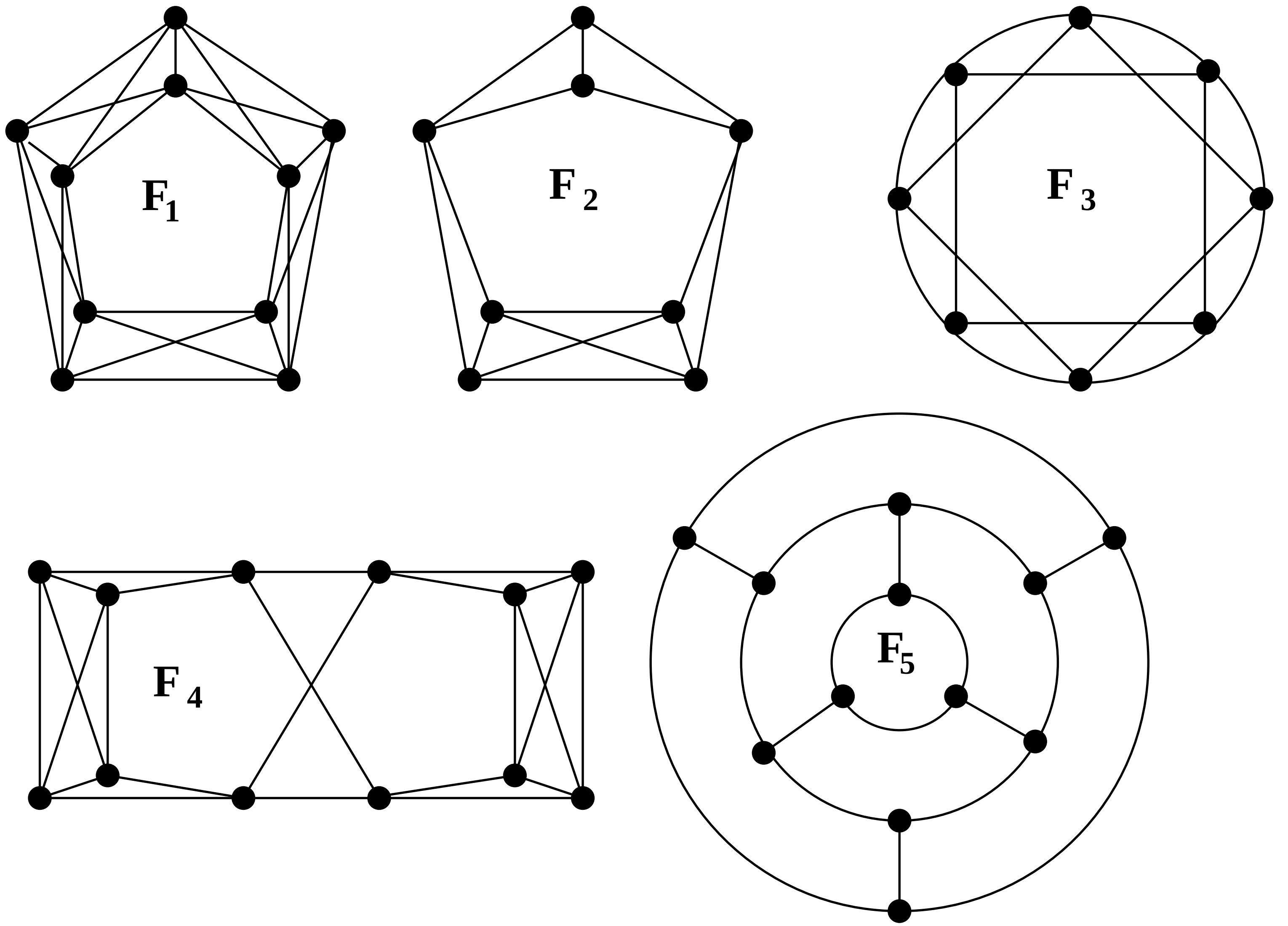
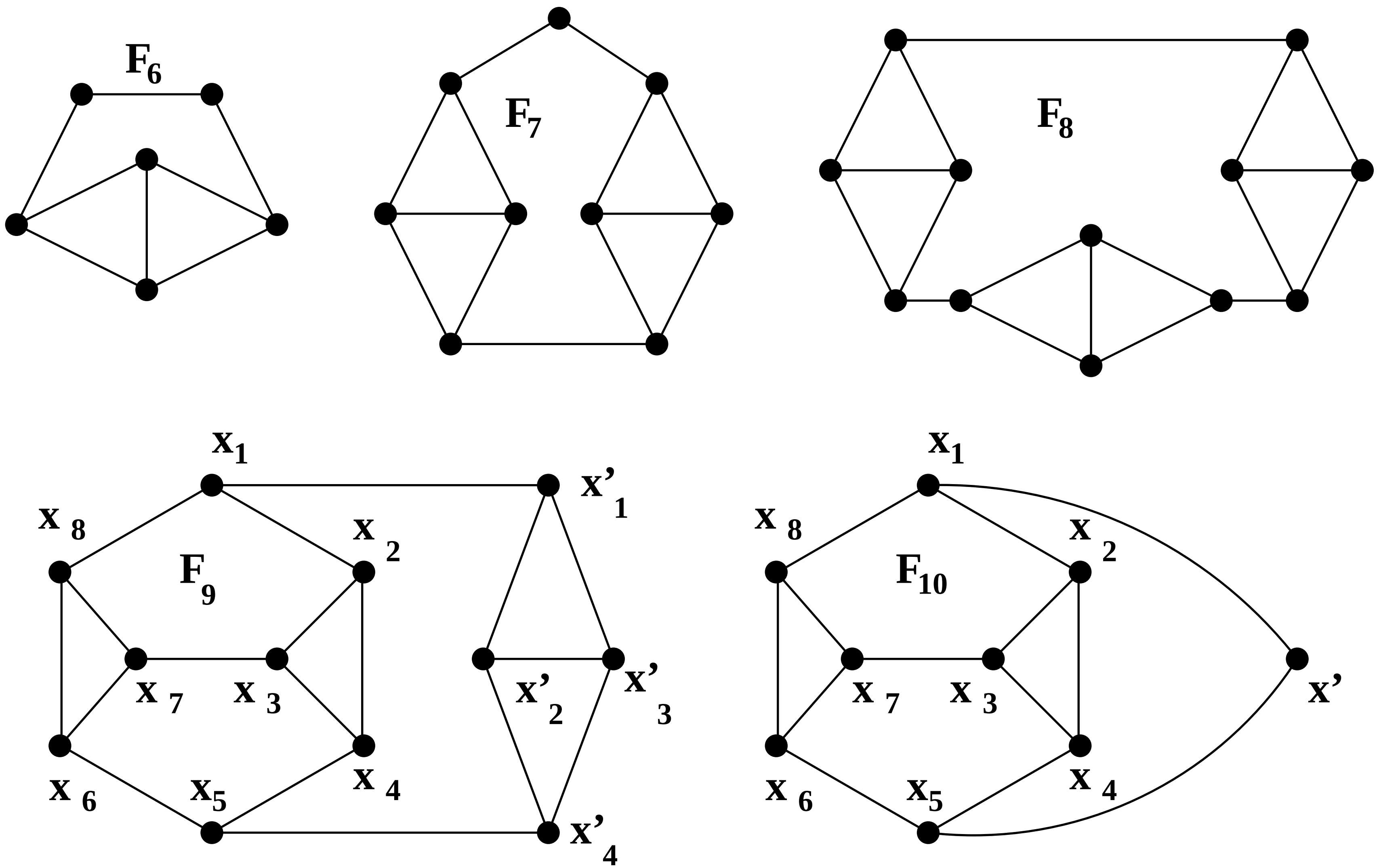
Clearly, a graph $G$ can be $r$-equitable only for one $r$. Call $G$ equitable if it is $r$-equitable for some $r$.

## Basic equitable graphs



# More basic equitable graphs



Conjecture 5. [K-K] If $r \geq 3$ is odd, then an $r$-colorable graph $G$ with $\Delta(G) \leq r$ does not have an equitable $r$-coloring if and only if a spanning subgraph of $G$ is the disjoint union of $K_{r,r}$ and basic $r$-equitable graphs.

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Theorem 12. [K-K] For every odd $r \geq 3$, if the Chen-Lih-Wu Conjecture holds for graphs $G$ with $\Delta(G) \leq r$, then Conjecture 5 holds for graphs $G$ with $\Delta(G) \leq r$.

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Corollary 13. [K-K] Conjecture 5 holds for $r = 3$.

Conjecture 6. [K-K] If $r \geq 3$, then an $r$-colorable graph $G$ with $\theta(G) \leq 2r$ does not have an equitable $r$-coloring if and only if a spanning subgraph of $G$ is the disjoint union of $K_{m,2r-m}$ for some odd $m$ and basic $r$-equitable graphs.

Conjecture 6. [K-K] If $r \geq 3$, then an $r$-colorable graph $G$ with $\theta(G) \leq 2r$ does not have an equitable $r$-coloring if and only if a spanning subgraph of $G$ is the disjoint union of $K_{m,2r-m}$ for some odd $m$ and basic $r$-equitable graphs.

Theorem 14. [K-K] For every $r \geq 3$, if Conjecture 4 holds for graphs $G$ with $\theta(G) \leq r$, then Conjecture 6 holds for graphs $G$ with $\theta(G) \leq r$.

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