The diameter game

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Joint work with

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Maker/Breaker games

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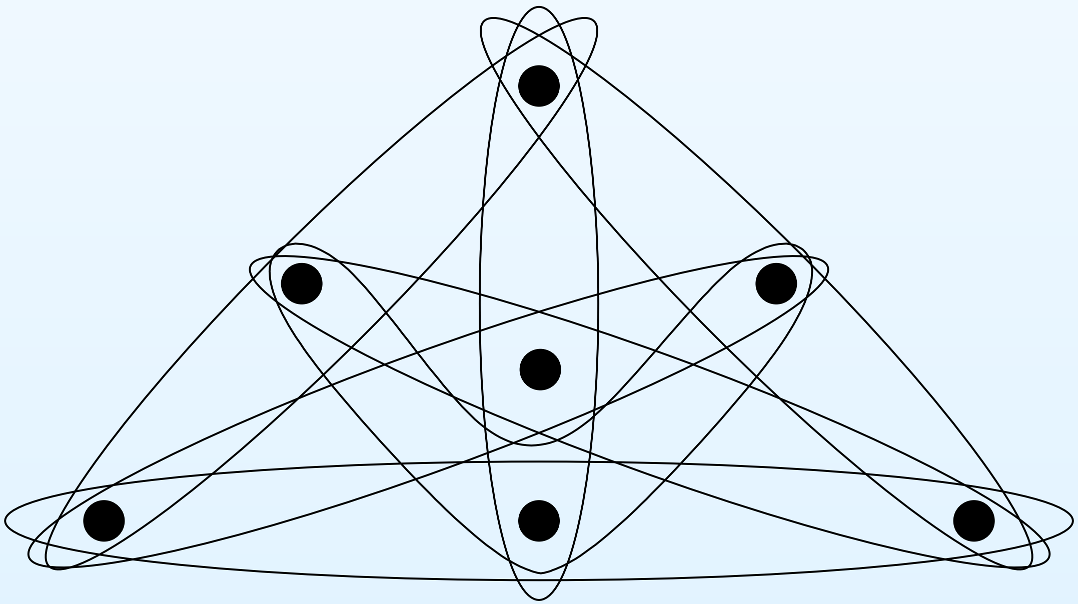

Maker/Breaker games

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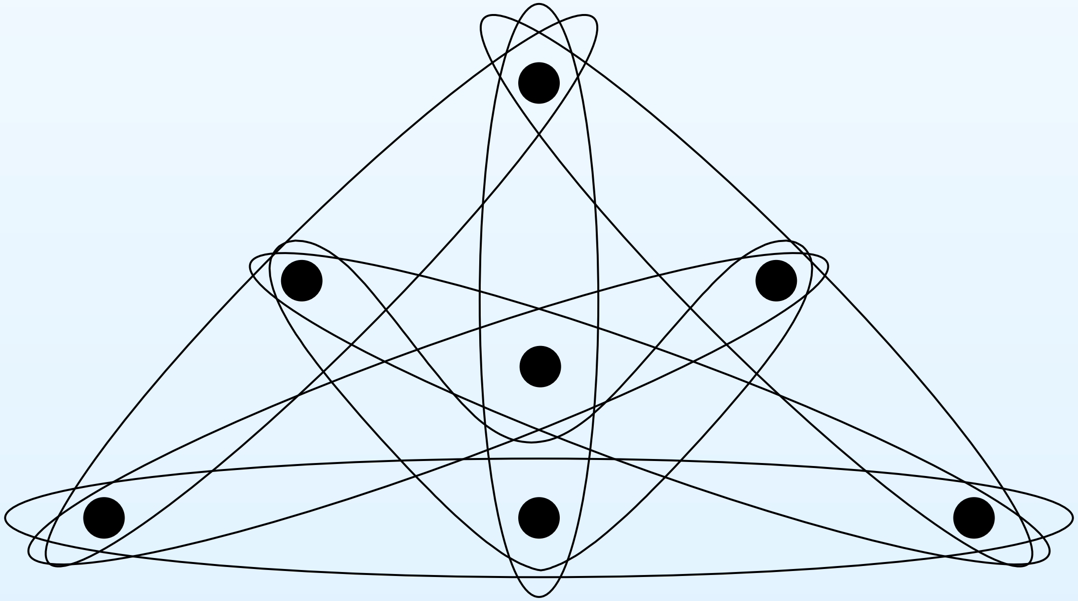

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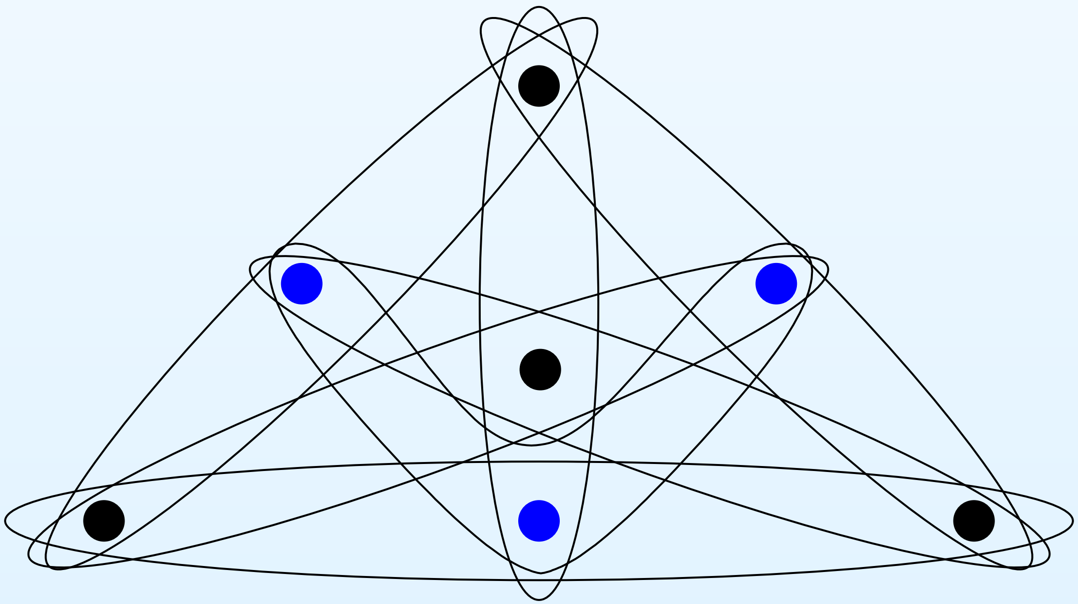
MAKER's goal is to occupy each vertex in some hyperedge.



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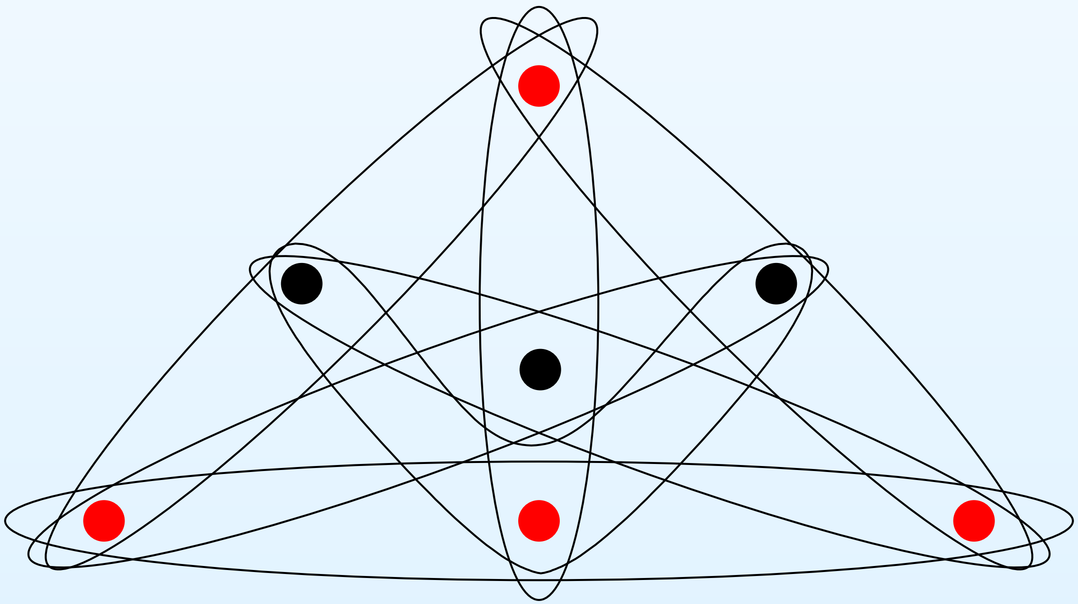
BREAKER's goal is to prevent this.



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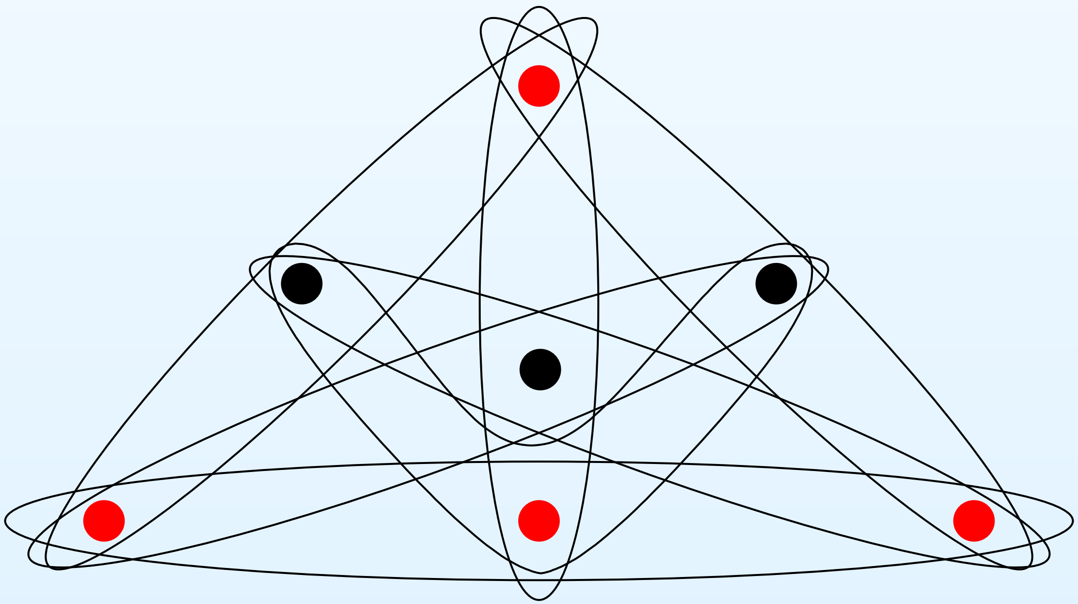

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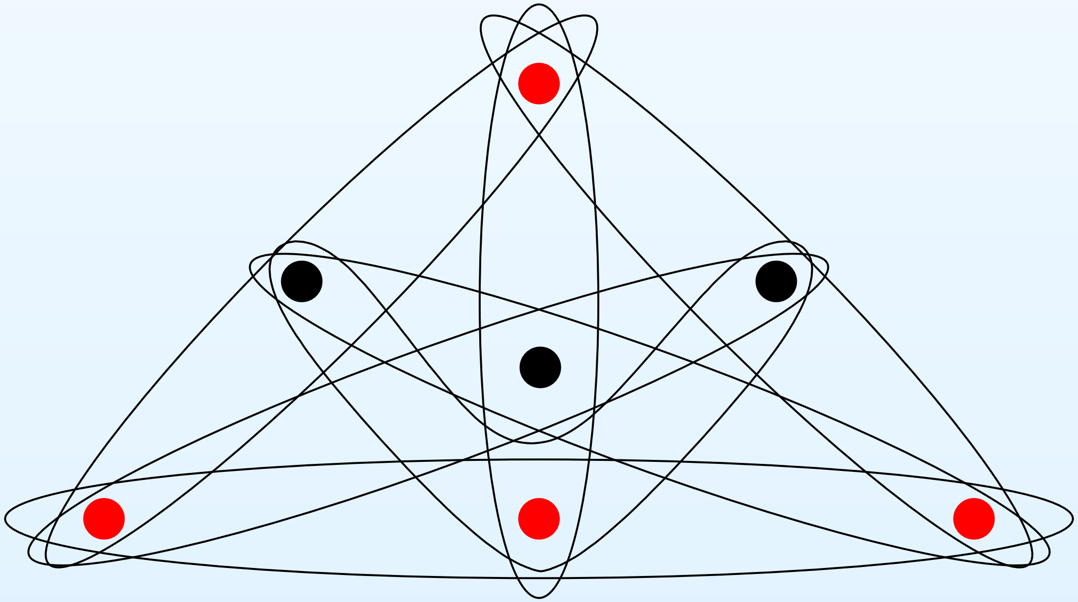
MAKER's goal is to occupy each vertex in some hyperedge.

BREAKER's goal is to prevent this; i.e., occupy at least one vertex from **each** hyperedge. There is no draw.



(1:1) diameter 2 game

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GRAPH GAMES:

MAKER and BREAKER in turns occupy edges of K_n.

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- MAKER gets to choose one edge not already taken.

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Most properties we want are monotone properties ($\mathcal{P}$).

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THE PROPERTY $\mathcal{P}$ GAME:

- MAKER wins if his graph has property $\mathcal{P}$.

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Does anyone have a winning strategy?

Winning the game

If $n \geq 4$, it doesn't matter who goes first!

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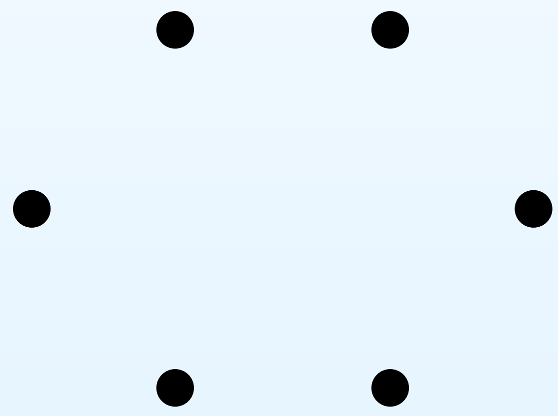

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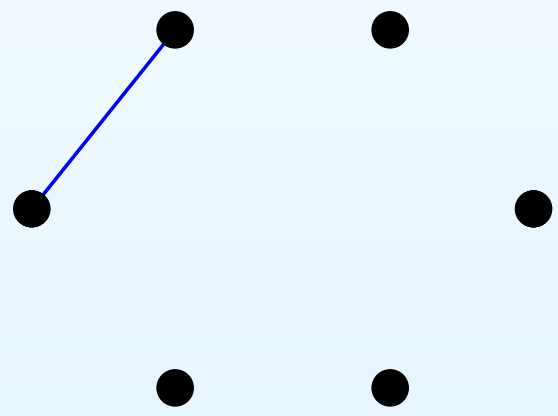

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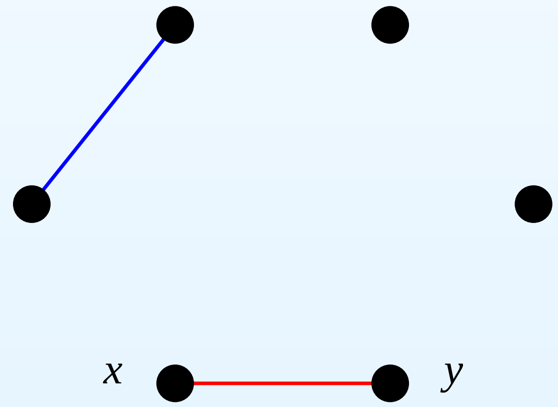

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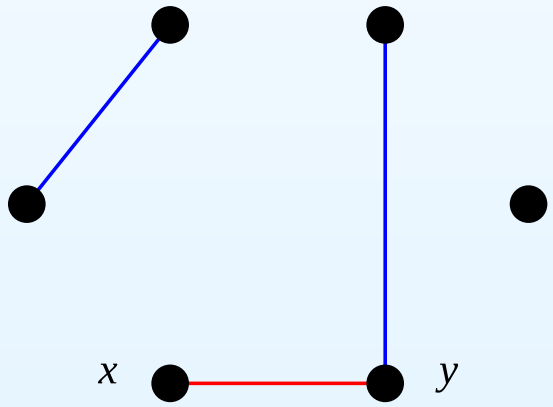
Then BREAKER plays a mimic strategy, keeping x and y apart.



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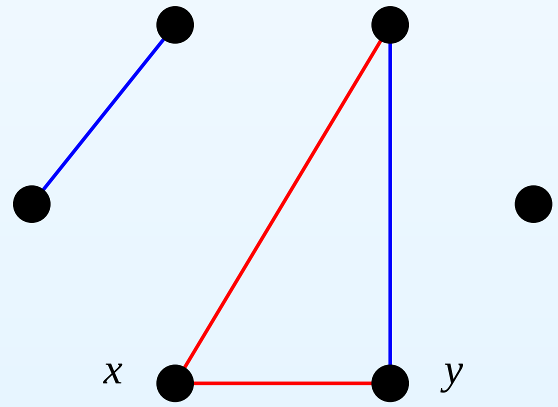

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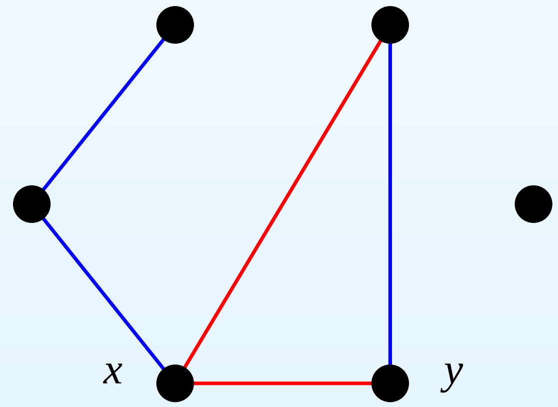

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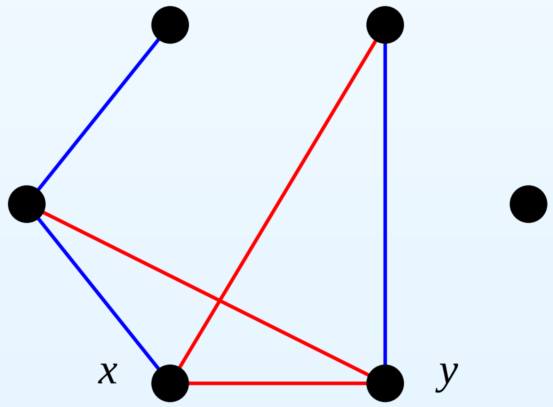

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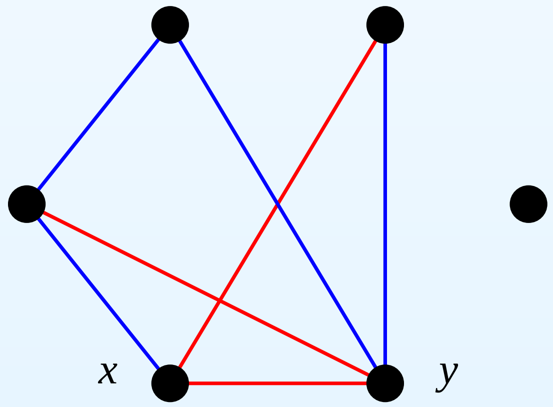

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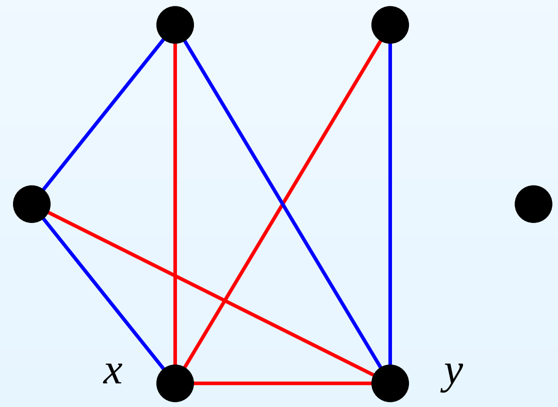

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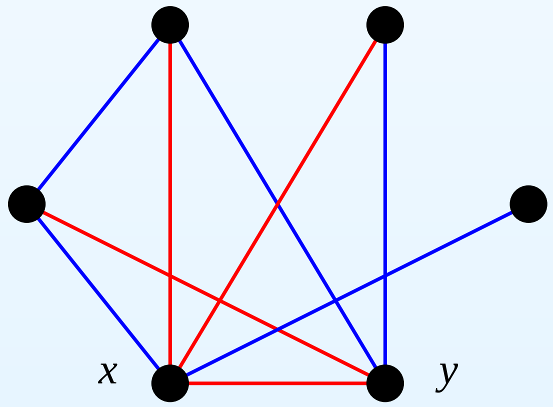

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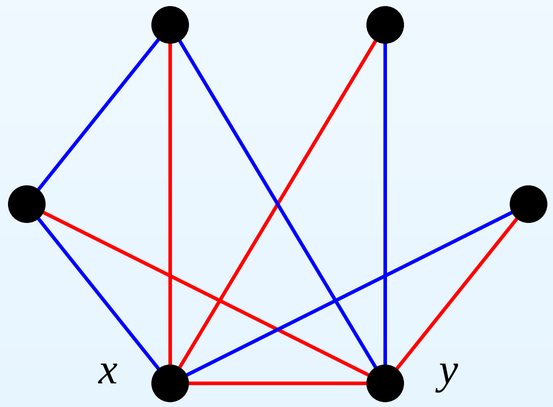

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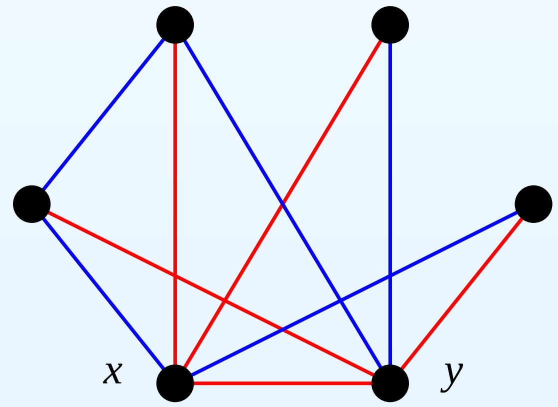

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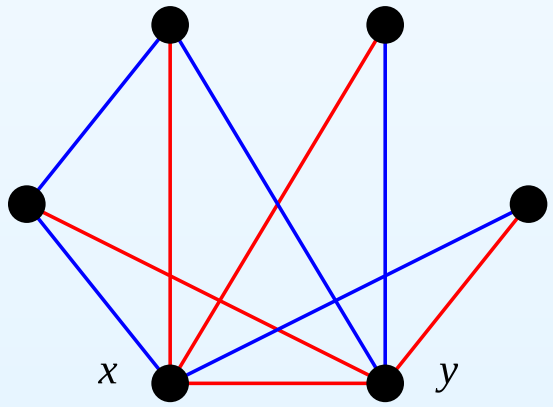

BREAKER wins! There is no way for MAKER to create a path of length at most 2 between x and y.

(a:b) games

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- FRIEZE-KRIVELEVICH-PIKHURKO-SZABÓ: MAKER can create a pseudo-random graph * if $a = b = 1$.

* Degrees approximately $n/2$, codegrees approximately $n/4$.

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For every monotone property $\mathcal{P}$, there is a threshold probability p_0 such that the random graph $G(n, p)$

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Let $p = \frac{a}{a+b}$, the proportion of edges belonging to MAKER when the game is over. In the connectivity and Hamiltonicity games, MAKER wins if $p \gg p_0$.

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In fact, the graph games all seem to follow probabilistic intuition.

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We know that $G(n,p)$ has diameter 2 (**whp**) if $p \gg \ln n/\sqrt{n}$.

But that says that $G(n,1/2)$ has diameter 2 and MAKER should win the (1:1)-game!

More strongly, Frieze, et al. said that MAKER can create a pseudo-random graph in the (1:1)-game. However, BREAKER can ensure that at least one pair is of distance at least 3.

Probabilistic Intuition Fails!

(a:b) diameter 2 game

What if $a > b$?

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I.e., if $b < a < \frac{n}{4 \ln n}$, then MAKER can ensure that his graph has minimum-degree at least $(n-1)/2$, if n is large enough.

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Consider the (2:2)-diameter 2 game.
This is called *accelerating the game*.

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The mimic strategy doesn't work, here.

Solution to (a:b) diameter 2 game

THEOREM.

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for some c, if n is big enough.

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The upper bound is close to probabilistic intuition.

Other results

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Let $d \geq 3$ and $a > 1$.

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MAKER wins the

- $\left(1 : c_1 n^{1-\lceil d/2 \rceil^{-1}} / \ln n\right)$-diameter d game.

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- $\left(1 : c_2 n^{1-(d-1)^{-1}}\right)$-diameter d game and the
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for absolute constants $c_1, c_2, c_3 > 0$, if n is big enough.

Diameter 3

For $d = 3$,

- MAKER wins the $\left(1 : O(n^{1/2}/\ln n)\right)$-diameter 3 game and
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But we can only guarantee that, for $a > 1$,

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Proof ideas

To show, for example, that Maker wins the $(2, n^{1/5})$-diameter two game, we divide Maker's strategy into two phases, playing alternating subgames in each phase.

For example: The subgames of Phase I:

- **Degree Game.** Maker can ensure that the minimum degree of his graph is $\frac{an}{a+b} - O\left(\sqrt{n \ln n}\right)$.

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- RATIO GAME. If BREAKER occupies many edges incident to a vertex, so does MAKER.
- CONNECTING HIGH VERTICES. Vertices with many incident edges chosen will connect to each other.

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Clearly, the existence of such a $\{T_i\}$ gives that BREAKER cannot win the degree game.

Thanks

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