

# Turán numbers of bipartite graphs under ear additions

Tao Jiang \*

Dept. of Mathematics and Statistics  
Miami University, Oxford, OH 45056, USA  
E-mail: jiangt@muohio.edu

Michael Salerno

Dept. of Mathematics and Statistics  
Miami University, Oxford, OH 45056, USA  
E-mail: salernmj@muohio.edu

## Abstract

The Turán number  $ex(n, H)$  of a graph  $H$  is the maximum number of edges in a graph of order  $n$  not containing a copy of  $H$ . Given a bipartite graph  $H$  and an edge  $uv$  in  $H$ , let  $L$  be obtained from  $H$  by adding a path of odd length  $l \geq 3$  between  $u$  and  $v$  via  $l - 1$  new vertices. We prove that  $ex(n, L) \leq 2ex(n, H) + O(n^{3/2})$ . Hence, if  $ex(n, H) = O(n^\alpha)$  for some  $\alpha \geq \frac{3}{2}$ , then  $ex(n, L) = O(n^\alpha)$  as well.

Let  $\mathcal{L}$  denote the class of bipartite graphs that can be obtained from a forest via a sequence of ear additions where each ear has odd length at least 3. Then  $\mathcal{L}$  is a subclass of 2-degenerate bipartite graphs. Our result implies that  $ex(n, H) = O(n^{3/2})$  holds for each  $H \in \mathcal{L}$ . Hence the well-known conjecture of Erdős and Simonovits [3] that every 2-degenerate bipartite graph  $H$  satisfies  $ex(n, H) = O(n^{3/2})$  holds for the subclass  $\mathcal{L}$ .

In this short note, we consider only simple finite graphs. We use standard graph theoretic notations as can be found in [10] for instance. Let  $H$  be a fixed graph. A classical problem in extremal graph theory is to determine the maximum number of edges in a graph on  $n$  vertices which does not contain a copy of  $H$ . This maximum value is the *Turán number* of  $H$ , and we will denote it by  $ex(n, H)$ .

For non-bipartite graphs  $H$ , the celebrated Erdős-Stone-Simonovits Theorem [4] determines  $ex(n, H)$  asymptotically, showing that  $ex(n, H) = (1 - \frac{1}{p-1})\binom{n}{2} + o(n^2)$ , where  $p = \chi(H)$ . The problem, however, is generally open when  $H$  is bipartite. In fact, even for some very small special bipartite graphs  $H$ , the order of magnitude of  $ex(n, H)$  is not yet determined.

Let  $Q_3$  denote the 3-dimensional cube. Erdős and Simonovits [5] showed that  $ex(n, Q_3) = O(n^{8/5})$ . Pinchasi and Sharir [9] later gave a new proof of the result. For the lower bound it is only known that  $ex(n, Q_3) \geq ex(n, C_4) = \Omega(n^{3/2})$ .

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Let  $k$  be a positive integer. A graph  $H$  is  $k$ -degenerate if its vertices can be linearly ordered such that each vertex has at most  $k$  neighbors appearing before it in the ordering. A fascinating conjecture by Erdős and Simonovits [6] states that every  $k$ -degenerate bipartite graph  $H$  satisfies  $ex(n, H) = O(n^{2-\frac{1}{k}})$ . Alon, Krivelevich, and Sudakov [1] showed that the conjecture holds for bipartite graphs in which all the vertices in one partite set have degree at most  $k$ . They noted that the result was implicit in a Füredi paper [8]. Much earlier, Erdős and Simonovits [5] made the conjecture for the special case when  $k = 2$ . Namely, they conjectured that every 2-degenerate bipartite graph  $H$  satisfies  $ex(n, H) = O(n^{3/2})$ . Even this weaker conjecture is still open.

Here, we consider a subclass of 2-degenerate bipartite graphs obtained as follows. Given a graph  $H$  and an edge  $uv$ , let  $L$  be obtained from  $H$  by adding a path of odd length  $l \geq 3$  between  $u$  and  $v$  via  $l - 1$  new vertices. We say that  $L$  is obtained from  $H$  via an  $l$ -ear addition to  $uv$ . Since  $l$  may vary, we drop the prefix  $l$  when more than one ear additions are performed. We show that any bipartite graph  $L$  that can be obtained from a forest via sequence of ear additions satisfies  $ex(n, L) = O(n^{3/2})$ . Obviously, the same holds for subgraphs of such a graph. Our method is based on the one used by Pinchasi and Sharir [9] in bounding  $ex(n, Q_3)$ . We will also need the simple fact that if  $T$  is a  $p$ -vertex tree then  $ex(n, T) \leq pn$ . Indeed, if  $G$  is an  $n$ -vertex graph with more than  $pn$  edges, then it contains a subgraph of minimum at least  $p$  and thus contains  $T$ .

**Theorem 1** *Let  $H$  be a bipartite graph on  $p$  vertices. Let  $l \geq 3$  be an odd integer. Let  $uv \in E(H)$ . Let  $L$  be obtained from  $H$  via an  $l$ -ear addition to  $uv$ . Then*

$$ex(n, L) \leq 2ex(n, H) + 6\sqrt{pl} \cdot n^{3/2}.$$

*Proof.* Let  $G$  be a graph on  $n$  vertices that does not contain a copy of  $L$ . We may assume that  $n \geq 5$  and  $p \geq 3$ . We prove that  $e(G) \leq 2ex(n, H) + 6\sqrt{pl} \cdot n^{3/2}$ . It is well known that  $G$  contains a bipartite subgraph  $F$  with  $e(F) \geq \frac{1}{2}e(G)$ . Let  $(X, Y)$  be a bipartition of  $F$ . For each edge  $e = xy$ , let  $F_e$  denote the subgraph of  $F$  induced by  $N_F(x) \cup N_F(y)$ . If  $F_e$  contains a path on at least  $pl$  vertices, we say that  $e$  is *heavy*; otherwise we say that  $e$  is *light*. Let  $F^*$  denote the spanning subgraph of  $F$  consisting only of heavy edges. Let  $\tilde{F}$  denote the spanning subgraph of  $F$  consisting only of light edges.

**Claim 1.** We have  $e(F^*) \leq ex(n, H)$ .

*Proof of Claim 1.* Suppose that  $e(F^*) > ex(n, H)$ , we derive a contradiction. By our assumption,  $F^*$  contains a copy  $H'$  of  $H$ , where say  $a$  and  $b$  are the images of  $u$  and  $v$ , respectively. Without loss of generality suppose  $a \in X$  and  $b \in Y$ . By our assumption  $F_{ab}$  has a path  $P$  on at least  $pl$  vertices. Since  $|V(P) \cap V(H')| \leq p - 2$ , it is easy to see that  $P - V(H')$  contains a path  $Q$  of length  $l - 2$ , say with endpoints  $c$  and  $d$ . Since  $l - 2$  is odd,  $c$  and  $d$  lie in different partite sets, say  $c \in Y$  and  $d \in X$ . Thus,  $c \in N_F(a)$  and  $d \in N_F(b)$  by our definition of  $F_{ab}$ . Now, we can extend  $H'$  to be a copy of  $L$  in  $F$  by adding to  $H'$  the path  $ac \cup Q \cup bd$ . Thus, in particular  $G$  contains a copy of  $L$ , a contradiction. ■

**Claim 2.** We have  $e(\tilde{F}) \leq 3\sqrt{pl} \cdot n^{3/2}$ .

*Proof of Claim 2.* For each edge  $e = xy \in E(\tilde{F})$ , let  $\tilde{F}_e$  denote the subgraph of  $\tilde{F}$  induced by  $N_{\tilde{F}}(x) \cup N_{\tilde{F}}(y)$ . Let  $N_e$  and  $E_e$  denote the number of vertices and the number of edges in

$\tilde{F}_e$ , respectively. By our assumption,  $e$  is light, and hence  $\tilde{F}_e$  contains no path on  $pl$  vertices. Thus,  $E_e \leq pl \cdot N_e$ . Summing over all  $e \in E(\tilde{F})$  we get

$$\sum_{e \in E(\tilde{F})} E_e \leq pl \cdot \sum_{e \in E(\tilde{F})} N_e. \quad (1)$$

For convenience, we call a  $P_3$  a *2-claw*. The middle vertex of a 2-claw will be called the *center* of the 2-claw. Let  $q$  denote the number of 2-claws in  $\tilde{F}$  and let  $S$  denote the number of  $C_4$ 's in  $\tilde{F}$ . Observe that

$$\sum_{e \in E(\tilde{F})} N_e = 2q \quad \text{and} \quad \sum_{e \in E(\tilde{F})} E_e = 4S. \quad (2)$$

Recall that  $(X, Y)$  is a bipartition of  $\tilde{F}$ . Each 2-claw in  $\tilde{F}$  has its center in either  $X$  or  $Y$ . Let  $W_X$  denote the number of 2-claws in  $\tilde{F}$  centered in  $X$  and  $W_Y$  the number of 2-claws in  $\tilde{F}$  centered in  $Y$ . Without loss of generality, we may assume that  $W_X \geq \frac{1}{2}q$ .

Let  $E = e(\tilde{F})$ . Note that  $\sum_{x \in X} d_{\tilde{F}}(x) = E$ , we have

$$\begin{aligned} q &\geq W_X = \sum_{x \in X} \binom{d_{\tilde{F}}(x)}{2} = \sum_{x \in X} \frac{d_{\tilde{F}}^2(x)}{2} - \sum_{x \in X} \frac{d_{\tilde{F}}(x)}{2} \\ &= \sum_{x \in X} \frac{d_{\tilde{F}}^2(x)}{2} - \frac{E}{2} \geq \frac{E^2}{2|X|} - \frac{E}{2} \geq \frac{E^2}{2n} - \frac{E}{2} \quad (\text{by convexity of } x^2). \end{aligned} \quad (3)$$

For each pair  $a, b \in Y$ , let  $W_{a,b}$  denote the number of common neighbors of  $a, b$  in  $\tilde{F}$  (which necessarily lie in  $X$ ). Observe that  $W_X = \sum_{a,b \in Y} W_{a,b}$  and  $S = \sum_{a,b \in Y} \binom{W_{a,b}}{2}$ . Now,

$$\begin{aligned} S &= \sum_{a,b \in Y} \binom{W_{a,b}}{2} = \sum_{a,b \in Y} \frac{W_{a,b}^2}{2} - \sum_{a,b \in Y} \frac{W_{a,b}}{2} = \sum_{a,b \in Y} \frac{W_{a,b}^2}{2} - \frac{W_X}{2} \\ &\geq \frac{W_X^2}{2\binom{|Y|}{2}} - \frac{W_X}{2} \geq \frac{W_X^2}{n^2} - \frac{W_X}{2} \quad (\text{by convexity of } x^2). \end{aligned} \quad (4)$$

Suppose  $\frac{W_X}{2} \geq \frac{W_X^2}{2n^2}$ . Then  $W_X \leq n^2$ . By (3), we have  $\frac{E^2}{2n} - \frac{E}{2} \leq n^2$ . From this, we get  $E \leq \frac{1}{2}(n + \sqrt{n^2 + 8n^3}) \leq 3n^{3/2} \leq 3\sqrt{pl} \cdot n^{3/2}$ , and we are done. Hence, we may assume  $\frac{W_X}{2} \leq \frac{W_X^2}{2n^2}$ . Then (3) yields

$$S \geq \frac{W_X^2}{2n^2}. \quad (5)$$

Thus, by (2) and  $W_X \geq \frac{q}{2}$  we have

$$\sum_{e \in E(\tilde{F})} E_e = 4S \geq \frac{2W_X^2}{n^2} \geq \frac{q^2}{2n^2}. \quad (6)$$

Now, by (1),(2), and (6), we have

$$\frac{q^2}{2n^2} \leq \sum_{e \in E(\tilde{F})} E_e \leq pl \cdot \sum_{e \in E(\tilde{F})} N_e = pl \cdot (2q).$$

From this, we get

$$q \leq 4pl \cdot n^2. \quad (7)$$

Combining (7) and (3), we get  $\frac{E^2}{2n} - \frac{E}{2} \leq 4pl \cdot n^2$ . Solving this for  $E$ , we get

$$e(\tilde{F}) = E \leq \frac{1}{2}(n + \sqrt{n^2 + 32pln^3}) \leq 3\sqrt{pl} \cdot n^{3/2}.$$

■

By Claim 1 and Claim 2, we have

$$e(G) \leq 2e(F) = 2[e(F^*) + e(\tilde{F})] \leq 2ex(n, H) + 6\sqrt{pl} \cdot n^{3/2}.$$

■

Let  $ex(n, n, H)$  denote the maximum number of edges in a bipartite graph  $G$  with  $n$  vertices in each partite set that does not contain a copy of  $H$ . Our proof of Theorem 1 in fact yields

**Proposition 2** *Let  $H$  be a bipartite graph on  $p$  vertices. Let  $l \geq 3$  be an odd integer. Let  $uv \in E(H)$ . Let  $L$  be obtained from  $H$  by an  $l$ -ear addition to  $uv$ . Then*

$$ex(n, n, L) \leq ex(n, n, H) + 3\sqrt{pl} \cdot n^{3/2}.$$

Theorem 1 readily imply the following.

**Corollary 3** *Let  $H$  be a bipartite graph. Let  $L$  be obtained from  $H$  via an  $l$ -ear addition to some  $uv \in E(H)$ , where  $l \geq 3$  is an odd integer. Then  $ex(n, L) = O(ex(n, H)) + O(n^{3/2})$ . Hence, in particular, if  $ex(n, H) = O(n^\alpha)$  for some  $\alpha \geq \frac{3}{2}$ , then  $ex(n, L) = O(n^\alpha)$ .*

Subdividing an edge  $uv$ , as usual, means replacing  $uv$  with a path  $uxv$  through a new vertex  $x$ . Since subdividing an edge  $l - 1$  times can be achieved by first attaching an ear of length  $l$  to that edge  $e$  and then deleting  $e$ , Corollary 3 implies

**Corollary 4** *Let  $H$  be a bipartite graph. Let  $L$  be obtained from  $H$  by subdividing some edge  $l - 1$  times, where  $l \geq 3$  is an odd integer. Then  $ex(n, L) = O(ex(n, H)) + O(n^{3/2})$ . Hence, in particular, if  $ex(n, H) = O(n^\alpha)$  for some  $\alpha \geq \frac{3}{2}$ , then  $ex(n, L) = O(n^\alpha)$ .*

Since for a forest  $F$ ,  $ex(n, F) = O(n)$ , repeated applications of Theorem 1 yield

**Corollary 5** *Let  $L$  be a bipartite graph obtained from a forest via a sequence of ear additions (where each ear has an odd length at least 3 and is attached to an existing edge). Then  $ex(n, L) = O(n^{3/2})$ .*

Corollary 5 allows us to construct many bipartite graphs  $L$  with  $ex(n, L) = O(n^{3/2})$ . They include, for instance, the  $s$  by  $t$  grid  $P_s \square P_t$ , the square book with  $t$  pages (formed by taking  $t$  copies of  $C_4$ 's sharing a common edge), and many others.

Observe that the graphs satisfying the condition of Corollary 5 and the subgraphs of such graphs form a subclass of 2-degenerate bipartite graphs. It is rather small. However, one interesting connection here is that a 2-degenerate bipartite graph may be viewed as obtained via sequence of adding an ear of length two to two vertices in the same partite set and then taking a subgraph of the resulting graph.

It would be interesting to study how the length of the added ear affects the Turán number of the new graph. It is natural to suspect that the longer the ear is, the less change in the Turán number it can cause. In particular, for all  $t \geq 2$ :

**Question 1** *Is it true that there is an absolute constant  $c$  such that if  $H$  is a bipartite graph and  $L$  is obtained from  $H$  by adding an ear of length at least  $2t - 1$  to an edge  $uv \in E(H)$  then  $ex(n, L) \leq c \cdot ex(n, H) + O(n^{1+1/t})$ ?*

A weaker question is as follows (with  $t \geq 2$ )

**Question 2** *Let  $L$  be a bipartite graph obtained from a forest by successively adding an ear of odd length at least  $2t - 1$  to an existing edge. Is it true that  $ex(n, L) = O(n^{1+1/t})$ ?*

Naturally, one could also modify both questions by requiring each added ear to have length exactly  $2t - 1$ . In general, it would be very interesting to further our study of bipartite graphs whose Turán numbers are on the order of  $n^{1+\epsilon}$  for small positive  $\epsilon$ . So far, results in this direction are very limited. Bondy and Simonovits showed [2] that  $ex(n, C_{2t}) = O(n^{1+1/t})$ . Later, Faudree and Simonovits [7] were able to extend this to show that  $ex(n, C_{k,t}) = O(n^{1+1/t})$ , where  $C_{k,t}$  is the graph formed by taking  $k$  pairwise internally disjoint paths of length  $t$  between two vertices. Our questions 1 and 2 are motivated by these earlier results.

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