

# Anti-Ramsey Numbers of Doubly Edge-Critical Graphs

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**Abstract:** Given a graph  $H$  and a positive integer  $n$ , *Anti-Ramsey number*  $AR(n, H)$  is the maximum number of colors in an edge-coloring of  $K_n$  that contains no polychromatic copy of  $H$ . The anti-Ramsey numbers were introduced in the 1970s by Erdős, Simonovits, and Sós, who among other things, determined

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this function for cliques. In general, few exact values of  $AR(n, H)$  are known. Let us call a graph  $H$  *doubly edge-critical* if  $\chi(H - e) \geq p + 1$  for each edge  $e \in E(H)$  and there exist two edges  $e_1, e_2$  of  $H$  for which  $\chi(H - e_1 - e_2) = p$ . Here, we obtain the exact value of  $AR(n, H)$  for any doubly edge-critical  $H$  when  $n \geq n_0(H)$  is sufficiently large. A main ingredient of our proof is the stability theorem of Erdős and Simonovits for the Turán problem. 2009 Wiley Periodicals, Inc. J Graph Theory 00: 1–9, 2009

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## 1. INTRODUCTION

We will use the standard notation of graph theory, which can be found e.g. in Bollobás' book [3].

A subgraph of an edge-colored graph is *rainbow* (or *polychromatic*) if all of its edges have different colors. Let  $\mathcal{F}$  be a family of graphs. For the purpose of this paper, we call an edge-coloring that contains no rainbow copy of any graph in  $\mathcal{F}$  an  $\mathcal{F}$ -free coloring. The *anti-Ramsey number*  $AR(n, \mathcal{F})$  is the maximum number of colors in an  $\mathcal{F}$ -free edge-coloring of  $K_n$ , the complete graph on  $n$  vertices. Anti-Ramsey numbers were introduced by Erdős et al. [8]. Various results about this extremal function have been obtained since then: [1, 10, 14, 11, 2, 12, 17, 15, 13, 4] to name a few.

Erdős et al. [8] showed that these numbers are very closely related to Turán numbers as follows.

Given a family  $\mathcal{F}$  of graphs, let us call a graph  $G$  containing no graph in  $\mathcal{F}$  as a subgraph an  $\mathcal{F}$ -free graph. Let  $EX(n, \mathcal{F})$  denote the set of  $\mathcal{F}$ -free graphs on  $n$  vertices. (When  $\mathcal{F}$  consists of a single graph  $F$ , we will write  $EX(n, F)$  instead of  $EX(n, \{F\})$ , etc.) The *Turán number*  $ex(n, \mathcal{F})$  is the maximum number of edges of a graph in  $EX(n, \mathcal{F})$ . Recall that the *Turán graph*  $T_{n,p}$  is a complete  $p$ -partite graph on  $n$  vertices in which each part has size  $\lfloor n/p \rfloor$  or  $\lceil n/p \rceil$ . Let  $t(n, p) = e(T_{n,p})$ . Turán [21] showed that  $ex(n, K_{p+1}) = t(n, p)$  and that  $T_{n,p}$  is the unique maximum graph in  $EX(n, K_{p+1})$ . More generally, the celebrated result of Erdős et al. [9, 7] states that for any  $\mathcal{F}$  we have

$$ex(n, \mathcal{F}) = t(n, p) + o(n^2) = \left(1 - \frac{1}{p}\right) \binom{n}{2} + o(n^2), \quad (1)$$

where  $p = \Psi(\mathcal{F})$  is the *subchromatic number*:

$$\Psi(\mathcal{F}) = \min\{\chi(F) : F \in \mathcal{F}\} - 1. \quad (2)$$

Let us give a few other definitions needed to state the results. The *derived family* of  $\mathcal{F}$  is

$$\mathcal{F}^- = \{H - e : H \in \mathcal{F}, e \in E(H)\}.$$

Let  $xy$  be an edge in a graph  $H$ . Let  $H_1, H_2$  be two copies of  $H - xy$ . For  $i = 1, 2$ , let  $x_i$  denote the image of  $x$  in  $H_i$  and  $y_i$  the image of  $y$  in  $H_i$ . Let  $H_{xy}^*$  denote the graph obtained by taking  $H_1$  and  $H_2$  and identifying  $x_1$  with  $x_2$  and  $y_1$  with  $y_2$ .

**Theorem 1** (Erdős et al. [8]). *Let  $\mathcal{F}$  be an arbitrary graph family. Let  $H \in \mathcal{F}$  and  $xy \in E(H)$  be such that  $\chi(H - xy) = \min\{\chi(F) : F \in \mathcal{F}^-\}$ . Then*

$$\text{ex}(n, \mathcal{F}^-) + 1 \leq \text{AR}(n, \mathcal{F}) \leq \text{ex}(n, H_{xy}^*).$$

*It follows from (1) that  $\text{AR}(n, \mathcal{F}) = t(n, p) + o(n^2)$ , where  $p = \Psi(\mathcal{F}^-)$ .*

**Proof.** We recall the proof because we will need it later.

Let  $G$  be a graph in  $\text{EX}(n, \mathcal{F}^-)$  of maximum size. Assign distinct colors to all of  $E(G)$  and a new color to all of  $E(\overline{G})$ . This yields an  $\mathcal{F}$ -free coloring of  $E(K_n)$ . So  $\text{AR}(n, \mathcal{F}) \geq \text{ex}(n, \mathcal{F}^-) + 1$ .

Next, suppose that  $c$  is an edge-coloring of  $K_n$  that uses more than  $\text{ex}(n, H_{xy}^*)$  colors. Let  $L$  be a *representing graph* of  $c$ , that is, it is a spanning subgraph of  $K_n$  obtained by taking one edge of each color in  $c$  (where  $L$  may contain isolated vertices). Then  $L$  must contain a copy  $F$  of  $H_{xy}^*$ . This is a rainbow copy of  $H_{xy}^*$  in  $c$ . Let  $H_1$  and  $H_2$  denote the two copies of  $H - xy$  in  $F$  as in the definition of  $H_{xy}^*$ . Note that  $H_1$  and  $H_2$  are edge-disjoint and since  $F$  is rainbow, no color appears in both  $H_1$  and  $H_2$ . Suppose without loss of generality that color  $c(xy)$  does not appear in  $H_1$ . Now  $H_1 \cup xy$  is a rainbow copy of  $H$ . This shows that  $\text{AR}(n, \mathcal{F}) \leq \text{ex}(n, H_{xy}^*)$ . ■

**Theorem 2** (Erdős et al. [8]). *For any  $p \geq 2$  and all sufficiently large  $n$ ,  $n \geq n_0(p)$ , we have*

$$\text{AR}(n, K_{p+2}) = t(n, p) + 1, \tag{3}$$

*and any coloring achieving this bound is obtained by taking a rainbow  $T_{n,p}$  and coloring all edges in its complement with the same (extra) color.*

Montellano-Ballesteros and Neumann-Lara [14] and independently Schiermeyer [17] showed that, in fact, (3) holds for every  $n \geq p + 1$  but did not characterize extremal colorings.

Let  $p$  be a positive integer. For  $0 \leq i \leq p$  and  $n \geq p + i$ , let  $T_{n,p}^i$  be obtained from the Turán graph by adding a single edge into each of the  $i$  largest vertex parts. (Thus,  $e(T_{n,p}^i) = t(n, p) + i$ .)

We say that a graph family  $\mathcal{F}$  is *doubly edge- $p$ -critical* if  $\Psi(\mathcal{F}^-) \geq p$  (i.e.  $\chi(H) \geq p + 1$  for every  $H \in \mathcal{F}^-$ ) but there exists a graph  $H \in \mathcal{F}$  and two edges  $e_1, e_2 \in E(H)$  such that  $\chi(H - e_1 - e_2) = p$ . In other words, we require that  $T_{n,p}^1$  is  $\mathcal{F}$ -free for every  $n$  but we can add two edges to some  $T_{n,p}$  (into two different parts or into the same part) and obtain an element of  $\mathcal{F}$  as a subgraph. Furthermore, we subdivide all such families into the following *types*: a doubly edge- $p$ -critical  $\mathcal{F}$  has *Type  $j$* , where  $j \geq 1$  is the maximum element of  $[p]$  such that  $T_{n,p}^j$  is  $\mathcal{F}$ -free for every  $n$ . This definition implies that if  $\mathcal{F}$  is of Type  $j$ , then, for every  $n \geq p + j$ , we have

$$\text{AR}(n, \mathcal{F}) \geq t(n, p) + j. \tag{4}$$

Indeed, the following coloring of  $K_n$  is  $\mathcal{F}$ -free and uses  $t(n, p) + j$  colors.

**Construction 3.** Take a copy  $H$  of  $T_{n,p}$ , color its edges with distinct colors and then color its complement  $\overline{H}$  with  $j$  new colors so that each of the  $p$  parts of  $H$  is monochromatic and such that each of these  $j$  new colors is used on at least one part.

The main result of our paper shows that this is best possible:

**Theorem 4.** Let  $p \geq 2$ . For an arbitrary (possibly infinite) doubly edge- $p$ -critical family  $\mathcal{F}$  of Type  $j$  there is an  $n_0$  such that for all  $n \geq n_0$  we have

$$\text{AR}(n, \mathcal{F}) = t(n, p) + j, \tag{5}$$

and all  $\mathcal{F}$ -free edge-colorings of  $K_n$  that achieve this bound are described in Construction 3.

This theorem generalizes Theorem 2 as  $K_{p+2}$  (more precisely,  $\{K_{p+2}\}$ ) is doubly edge- $p$ -critical of Type 1 for  $p \geq 2$ . One can easily generate many examples of doubly edge-critical graphs. One is  $T_{n,p}^2$ . Further examples can be obtained by taking the disjoint union of two edge- $p$ -critical graphs. (A graph  $H$  is *edge- $p$ -critical* if for some edge  $e \in E(H)$  we have  $\chi(H - e) = p < \chi(H)$ .) Also, one can start with an arbitrary doubly edge- $p$ -critical graph  $H$  and add a new set  $A$  of at least 2 vertices so that  $A$  spans no edge while we have a complete bipartite graph between  $V(H)$  and  $A$ . Some variations of the last construction are possible: for example, one can let  $A$  span certain graphs  $F$  although not every  $F$  would do here (e.g.  $F = K_1$ , that is,  $|A| = 1$ , does not work for some  $H$ ).

If  $\mathcal{F}$  is doubly edge- $p$ -critical, then we can choose a graph  $H \in \mathcal{F}^-$  in (3) which is edge- $p$ -critical. An important result of Simonovits [18] states that  $\text{ex}(n, H) = t(n, p)$  for all large  $n$ . This was one of our motivations: to prove the analog of Simonovits' result to anti-Ramsey numbers. Our result is a bit more complicated to prove (and to state) but the main idea is the same: we obtain the exact result using the stability approach. The proof can be found in Section 3.

An unpublished conjecture of the first author states that  $\text{AR}(n, \mathcal{F}) \leq \text{ex}(n, \mathcal{F}^-) + O(n)$  for any family  $\mathcal{F}$  (see [10] for some discussions). Our Theorem 4 confirms the conjecture in the case when, for some  $p \geq 2$ ,

$$\text{ex}(n, \mathcal{F}^-) = t(n, p) \quad \text{for all } n \geq n_0(p, \mathcal{F}). \tag{6}$$

Indeed, any  $\mathcal{F}^-$  satisfying (6) has subchromatic number  $p$  while some  $T_{N,p}^1$  is not  $\mathcal{F}^-$ -free (for otherwise  $\text{ex}(n, \mathcal{F}^-) \geq t(n, p) + 1$  for all  $n$ ). This shows that  $\mathcal{F}$  is doubly edge- $p$ -critical. It is possible that our method might apply in some other cases when  $\text{ex}(n, \mathcal{F}^-)$  can be determined via the stability approach. However, it is not clear how to proceed in general.

## 2. PRELIMINARIES

Here, we present some results and definitions that are needed for our proof of Theorem 4 in Section 3.

For a set  $X$  and an integer  $k$ , let  $\binom{X}{k} = \{A \subseteq X : |A| = k\}$ . Given a graph  $G$  and disjoint subsets  $A_1, \dots, A_p$  of vertices in  $G$ , we use  $G(A_1, \dots, A_p)$  to denote the  $p$ -partite subgraph of  $G$  with parts  $A_1, \dots, A_p$  containing all the edges of  $G$  connecting different  $A_i$ 's. Let  $K(A_1, \dots, A_p)$  denote the complete  $p$ -partite graph with parts  $A_1, \dots, A_p$ . If  $S \subseteq V(G)$ , then  $G[S]$  denotes the subgraph of  $G$  induced by  $S$ .

**Lemma 5.** *Let  $N, m, p \geq 2$  and  $s \geq 0$  be integers. Let  $G$  be a  $p$ -partite graph with parts  $A_1 \cup S, A_2, \dots, A_p$  where  $|S| = s$ ,  $|A_i| = N$  for  $i = 1, \dots, p$ , and  $S, A_1, \dots, A_p$  are pairwise disjoint. Suppose that there is no selection of subsets  $D_1 \subseteq A_1, D_2 \subseteq A_2, \dots, D_p \subseteq A_p$  with  $|D_i| = m$  for each  $i$  such that  $G[D_1 \cup S, D_2, \dots, D_p]$  is a complete  $p$ -partite graph. Let  $K = K(A_1 \cup S, A_2, \dots, A_p)$ . Let  $E^*(S)$  denote the set of edges of  $K$  incident to  $S$ . Then*

1.  $|E(K) \setminus E(G)| \geq \frac{1}{2} \binom{N}{m}^2$ , or
2.  $|(E(K) \setminus E(G)) \cap E^*(S)| \geq \frac{1}{2} \binom{N}{m}$ .

**Proof.** There are  $M_1 = \binom{N}{m}^p$  ways to select the  $D_i$ 's. By our assumption, for each selection, the  $p$ -partite graph  $G' = G[D_1 \cup S \cup D_2 \cup \dots \cup D_p]$  misses some edge of  $K$  whose two endpoints are in  $D_1 \cup S \cup D_2 \cup \dots \cup D_p$ . Suppose first that for at least  $\frac{1}{2}M_1$  of these selections,  $G'$  misses an edge not incident to  $S$ . For each edge  $xy \in (E(K) \setminus E(G)) \setminus E^*(S)$ , the number ways to select the  $D_i$ 's such that  $x, y \in D_1 \cup \dots \cup D_p$  is  $M_2 = \binom{N}{m}^{p-2} \binom{N-1}{m-1}^2$ . So, we have

$$|E(K) \setminus E(G)| \geq |(E(K) \setminus E(G)) \setminus E^*(S)| \geq \frac{(1/2)M_1}{M_2} = \frac{1}{2} \binom{N}{m}^2.$$

For the rest of the proof, suppose that for at least  $\frac{1}{2}M$  of the ways to select the  $D_i$ 's,  $G'$  misses an edge of  $E(K) \setminus E(G)$  that is incident to  $S$ . Fixing an edge  $xy \in (E(K) \setminus E(G)) \cap E^*(S)$ , where  $x \in S$ , the number of ways to select the  $D_i$ 's such that  $y \in D_2 \cup \dots \cup D_p$  is  $M_3 = \binom{N}{m}^{p-1} \binom{N-1}{m-1}$ . This implies that

$$|(E(K) \setminus E(G)) \cap E^*(S)| \geq \frac{(1/2)M_1}{M_3} = \frac{1}{2} \binom{N}{m}. \quad \blacksquare$$

Also, we will need the following very useful result.

**Theorem 6** (The Stability Theorem, Erdős [5, 6], Simonovits [18]). *Let  $\mathcal{F}$  be a family of graphs with subchromatic number  $p$ . For every  $\varepsilon > 0$ , there exist  $\delta > 0$  and  $n_\varepsilon$  such that for  $n > n_\varepsilon$  if  $G$  is a graph in  $\text{EX}(n, \mathcal{F})$  with*

$$e(G) > \text{ex}(n, \mathcal{F}) - \delta n^2,$$

*then  $G$  can be obtained from  $T_{n,p}$  by changing at most  $\varepsilon n^2$  edges.*

### 3. PROOF OF THEOREM 4

Let  $\mathcal{F}$  be a doubly edge- $p$ -critical family of graphs. We have already demonstrated the lower bound in (5). Let us prove the upper bound and characterize the extremal colorings.

Fix a large integer  $m$ , a graph  $H \in \mathcal{F}$ , and edges  $e_1, e_2 \in E(H)$  with  $H - e_1 - e_2 \subseteq T_{pm,p}$ . Let  $\mathcal{F}$  be of Type  $j$ . If  $j < p$ , then (enlarging  $m$  if necessary) fix  $F \in \mathcal{F}$  such that  $F \subseteq T_{pm,p}^{j+1}$ . If  $j = p$ , then we can let e.g.  $F = H$ .

Given  $F, H$ , and  $m$ , choose, in this order, small positive constants

$$\frac{1}{m} \gg \varepsilon \gg \delta \gg \frac{1}{n_0},$$

where  $\alpha \gg \beta$  means that  $\beta$  is sufficiently small depending on  $\alpha$ . Let  $n \geq n_0$  be arbitrary. We do not specify the dependencies between constants explicitly, since we use Theorem 6 as a “black box.” In particular, we make no attempt to optimize the constants.

Let  $c$  be an extremal  $\mathcal{F}$ -free edge-coloring of the host graph  $K_n$  on the vertex-set  $V$ . Let  $L$  be a representing graph of  $c$ , that is, it is obtained by picking one edge from each color class. By the extremality of  $c$  we have

$$e(L) \geq t(n, p) + j. \tag{7}$$

By the proof of Theorem 1,  $L$  cannot contain  $H_{xy}^*$ . Since  $e(L) > t(n, p) > ex(n, H_{xy}^*) - \delta n^2$ , by Theorem 6  $L$  can be obtained from  $T_{n,p}$  by changing at most  $\varepsilon n^2$  edges.

Among all complete  $p$ -partite graphs with vertex-set  $V$ , choose  $T$  so that  $|E(T) \cap E(L)|$  is maximum. Let the graph  $T'$  on  $V$  have  $E(T) \cap E(L)$  as the edge-set. Our choice of  $T$  implies that

$$e(T') \geq t(n, p) - \varepsilon n^2. \tag{8}$$

Since  $e(T) \leq t(n, p)$ , we have

$$|E(T) \setminus E(L)| \leq \varepsilon n^2. \tag{9}$$

Let  $A_1, A_2, \dots, A_p$  denote the parts of  $T$ . We have, for example,  $|A_i| \geq n/2p$  for all  $i \in [p]$ , for otherwise it is easy to see that the imbalance of the  $|A_i|$ 's forces  $e(T) < t(n, p) - \varepsilon n^2$ , contradicting (8).

Suppose first that for each  $i = 1, \dots, p$ , only one color is used inside  $A_i$  in the coloring  $c$ .

We apply Lemma 5 with  $S = \emptyset$  to  $T'$  (or rather to an arbitrary equipartite subgraph of  $T'$  with each part having size  $N = \lfloor n/2p \rfloor$ ). By (9), this gives a complete  $p$ -partite subgraph  $K(D_1, \dots, D_p) \subseteq T'$  with  $|D_i| \geq m + p$  for each  $i \in [p]$ . (Note that we apply Lemma 5 with respect to  $m + p$  rather than  $m$ .) Let  $c_i$  be the common color of  $\binom{D_i}{2}$ . Suppose there are exactly  $r \leq p$  different elements in  $\{c_1, \dots, c_p\}$ . By removing  $p$  vertices from each  $D_i$ , we can find a new complete  $p$ -partite graph  $K' = T'[D_1 \cup \dots \cup D_p]$  such that  $c(E(K')) \cap \{c_1, \dots, c_p\} = \emptyset$ . It follows that  $r \leq j$  for otherwise  $j < p$  and  $G$  contains a rainbow  $T_{mp,p}^{j+1}$ , and thus a rainbow  $F$ , a contradiction.

Now, (7) implies that  $r = j$ ,  $e(T) = t(n, p)$  (and so  $T \cong T_{n,p}$ ), and the coloring  $c$  is indeed described by Construction 3.

So, let us assume that some part  $A_i$  has at least 2 different colors. We will derive a contradiction. The following observation will be useful in our quest.

**Claim 1.** *There is no complete  $p$ -partite graph  $K(D_1, \dots, D_p) \subseteq T'$  such that  $|D_i| \geq m+7$  for each  $i \in [p]$  and  $c(uv) \neq c(xy)$  for some  $uv, xy$  in one part  $D_i$ .*

**Proof of Claim.** Suppose that the claim is false. Without loss of generality,  $u, v, x, y \in D_1$ . We can assume that the pairs  $uv$  and  $xy$  are disjoint. (Otherwise pick a third pair disjoint from both  $uv$  and  $xy$ ; its color is different from at least one of  $c(uv)$  or  $c(xy)$ .)

By removing 6 vertices from each  $D_i$ , we can make sure that no color present in  $\binom{\{u,v,x,y\}}{2}$  appears in the new graph  $K' = K(D_1, \dots, D_p)$  (while  $u, v, x, y$  are still in  $D_1$ ). The graph  $H \in \mathcal{F}$  can be embedded into  $T_{mp,p} + f_1 + f_2$ , where  $f_1, f_2$  are two extra edges.

Suppose first that  $f_1, f_2$  are in one part of the Turán graph. If they are disjoint, then we can map  $f_1, f_2$  to  $uv, xy$  and extend this mapping to an arbitrary embedding  $T_{mp,p} \subseteq K'$ . This gives a rainbow copy of  $H$ , a contradiction. If  $f_1 \cap f_2 \neq \emptyset$ , then we can map  $f_1$  to  $ux$  and  $f_2$  to one of  $uv, xy$ , namely to a pair whose color is different from  $c(ux)$ . Again this gives us a rainbow  $H$  because we took care to eliminate any color in  $\binom{\{u,v,x,y\}}{2}$  from  $E(K')$ .

So let us assume that  $f_1, f_2$  belong to different parts of  $T_{mp,p}$ . Let us map  $f_2$  to an arbitrary pair  $wz$  in  $D_2$ . If  $c(wz) \in \{c(uv), c(xy)\}$ , then map  $f_1$  to one of  $uv, xy$ , namely to a pair whose color is different from  $c(wz)$ . Note that the  $c$ -colors of the images of  $f_1$  and  $f_2$  do not appear on  $E(K')$ , so we can extend this to a rainbow copy of  $H$ . If  $c(wz) \notin \{c(uv), c(xy)\}$ , then at most one edge between  $wz$  and  $\{u, v, x, y\}$  can have  $c$ -color  $c(wz)$ , and we can safely map  $f_1$  either to  $uv$  or  $xy$ . At most one edge  $f$  of  $L$  can have color  $c(wz)$ . Since we still have some freedom (each current  $|D_i| \geq m+1$ ) we can find a copy of  $H$  avoiding  $f$ . This  $H$ -subgraph is rainbow, contradicting our assumptions. This finishes the proof of the claim. ■

Call the edges in  $E(T) \setminus E(L)$  *missing* and the edges in  $E(L) \setminus E(T)$  *bad*. Let  $M$  (resp.  $B$ ) be the graph on  $V$  consisting of all missing (resp. bad) edges. Since  $e(L) \geq e(T)$ , we have

$$e(B) \geq e(M). \tag{10}$$

Let  $\gamma = 1/(25pm)$  and  $W = \{x : d_M(x) \geq \gamma n\}$ .

**Claim 2.** *If  $uv, xy$  are any two edges (possibly intersecting) with different colors inside some  $A_i$ , then  $W \cap \{u, v, x, y\} \neq \emptyset$ .*

**Proof of Claim.** For notational convenience, suppose  $i = 1$ . Let  $S = \{u, v, x, y\}$ . Let  $C_1$  be a subset of  $A_1 \setminus S$  of size  $N = \lfloor n/3p \rfloor$ ;  $C_1$  exists since  $|A_1| \geq n/2p$ . For  $i = 2, \dots, p$ , let  $C_i$  be a subset of  $A_i$  of size  $N$ .

By Claim 1, we cannot find  $(m+7)$ -sets  $D_i \subseteq C_i$ ,  $i \in [p]$ , such that  $L(D_1 \cup S, D_2, \dots, D_p)$  is a complete  $p$ -partite graph. By (9) and Lemma 5,  $S$  is incident to at least  $N/(2(m+7))$  missing edges. Thus at least one vertex of  $S$  is incident to at least  $N/(8(m+7)) \geq \gamma n$  edges, as required. ■

Claim 2 in particular implies that  $e(M) \geq \gamma n$ . Hence,  $e(B) \geq \gamma n$ . Edges in  $B$  have endpoints in the same  $A_i$ . Without loss of generality, suppose  $B[A_1]$  contains the maximum number of edges of  $B$  among all  $A_i$ 's. Then  $e(B[A_1]) \geq e(B)/p \geq \gamma n/p \geq 2$ . Let  $W_1 = W \cap A_1$ . Since  $B[A_1]$  has at least two edges (and all edges of  $B \subseteq L$  have different colors), we have  $W_1 \neq \emptyset$  by Claim 2. Again, by Claim 2, at most one edge in  $B[A_1]$  can be disjoint from  $W_1$ . Therefore, the edges in  $B[A_1]$  can be covered by at most  $|W_1| + 1$  vertices in  $A_1$ . Let  $x \in A_1$  be such that  $d_B(x) = \max_{u \in A_1} d_B(u)$ . Then  $e(B[A_1]) \leq (|W_1| + 1) \cdot d_B(x)$ . On the other hand, there are at least  $|W_1| \cdot \gamma n$  edges in  $M$  incident to  $W_1$ . So,  $e(M) \geq |W_1| \cdot \gamma n$ . By our assumption,  $e(B[A_1]) \geq e(B)/p$ . So,

$$|W_1| \cdot \gamma n \leq e(M) \leq e(B) \leq e(B[A_1]) \cdot p \leq (|W_1| + 1) \cdot d_B(x) \cdot p.$$

Let  $\alpha = \gamma/2p$ . We get

$$d_B(x) \geq \frac{|W_1|}{|W_1| + 1} \cdot \frac{\gamma n}{p} \geq \frac{\gamma n}{2p} = \alpha n.$$

In other words,  $|N_L(x) \cap A_1| \geq \alpha n$ , where  $N_L(x)$  denotes the set of neighbors of  $x$  in the graph  $L$ . If for some  $i$ ,  $|N_L(x) \cap A_i| < |N_L(x) \cap A_1|$ , then by moving  $x$  from  $A_1$  to  $A_i$  we would strictly increase  $e(T')$ , contradicting our choice of  $T$ . Hence, for each  $i = 1, \dots, p$ ,  $|N_L(x) \cap A_i| \geq \alpha n$ . For each  $i = 1, \dots, p$ , let  $C_i$  be a subset of  $A_i \cap N_L(x)$  of size  $N = \lfloor \alpha n \rfloor$ .

Note that  $L(C_1, \dots, C_p)$  cannot contain a copy of  $T_{(m+7)p,p}$ . Indeed, since  $x$  is connected by an  $L$ -edge to every vertex of  $C_1 \cup \dots \cup C_p$ , we can swap  $x$  with some vertex of  $T_{(m+7)p,p}$  in a way that contradicts Claim 1. Applying Lemma 5, with  $G = L(C_1, \dots, C_p)$ ,  $K = T(C_1, \dots, C_p) \subseteq T$ ,  $N = \lfloor \alpha n \rfloor$ ,  $S = \emptyset$ , we conclude that

$$|E(T(C_1, \dots, C_p)) \setminus E(L(C_1, \dots, C_p))| \geq \frac{1}{2} \left( \frac{N}{m+7} \right)^2 > \epsilon n^2,$$

which contradicts (9). This completes the proof of Theorem 4. ■

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Q3

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