



# Canonical Ramsey numbers and properly colored cycles

Tao Jiang

Department of Mathematics and Statistics, Miami University, Oxford, OH 45056, USA

## ARTICLE INFO

### Article history:

Received 29 September 2008

Received in revised form 24 December 2008

Accepted 30 December 2008

Available online 21 January 2009

### Keywords:

Canonical Ramsey

Properly colored

Cycle

Rainbow

Color degree

## ABSTRACT

We improve the previous bounds on the so-called *unordered Canonical Ramsey numbers*, introduced by Richer [D. Richer, Unordered canonical Ramsey numbers, Journal of Combinatorial Theory Series B 80 (2000) 172–177] as a variant of the canonical Ramsey numbers introduced by Erdős and Rado [P. Erdős, R. Rado, A combinatorial theorem, Journal of the London Mathematical Society 25 (4) (1950) 249–255].

Then we prove a conjecture raised by Axenovich, Jiang, and Tuza in [M. Axenovich, T. Jiang, Zs. Tuza, Local anti-Ramsey numbers of graphs, Combinatorics, Probability, and Computing 12 (2003) 495–511], showing that for each  $k \geq 4$  and large  $n$ , every edge-coloring of  $K_n$  in which at least  $256k$  different colors appear at each vertex contains a properly colored cycle of length exactly  $k$ . Here, a cycle is *properly colored* if no two incident edges in it have the same color. The bound  $256k$  is tight up to a constant factor.

© 2009 Elsevier B.V. All rights reserved.

## 1. Introduction

We consider only simple finite graphs. We use standard graph theoretic terms as can be found in [8]. Given a positive integer  $n$ , we use  $[n]$  to denote  $\{1, \dots, n\}$ . In classical Ramsey theory, we fix the number of colors in an edge-coloring of  $K_N$  and study when a given monochromatic subgraph is forced when  $N$  increases. When one allows an arbitrary number of colors to be used, one can naturally no longer force a given monochromatic subgraph no matter how large  $N$  is. However, the celebrated Canonical Ramsey Theorem of Erdős and Rado [4] shows that one of the four special color patterns will still be forced. We paraphrase their theorem as follows. When the endpoints of  $x$  and  $y$  of an edge  $e$  are integers, we call  $\max\{x, y\}$  the *higher endpoint* and  $\min\{x, y\}$  the *lower endpoint*.

**Theorem 1** (Erdős-Rado [4]). *Let  $p$  be a positive integer. Then there exists a least positive integer  $N = er(p)$  such that if the edges of the complete graph  $K_N$  with vertex set  $\{1, \dots, N\}$  are colored using an arbitrary number of colors, then there exists a complete subgraph with  $p$  vertices on which the coloring is of one of the four canonical types:*

1. *rainbow* – no two edges have the same color;
2. *monochromatic* – all edges have the same color;
3. *upper lexical* – two edges have the same color if and only if they have the same higher endpoint;
4. *lower lexical* – two edges have the same color if and only if they have the same lower endpoint.

The best known estimates of  $er(p)$  are due to Lefmann and Rödl [5] who showed that there exist constants  $c, c'$  such that for every positive integer  $p$ ,  $2^{cp^2} \leq er(p) \leq 2^{c'p^2 \log p}$ .

Note that in the Canonical Ramsey Theorem, the vertices of  $K_N$  are preordered. Some variations of the Canonical Ramsey Theorem have been considered where the vertices of  $K_N$  are not preordered. Then the last two types: upper lexical and lower lexical colorings can be naturally combined into the so-called lexical coloring. An edge-coloring  $c$  of a graph  $G$  is *lexical* if the vertices of  $G$  can be ordered so that two edges have the same color if and only if they share the same lower endpoint.

E-mail address: [jiangt@muohio.edu](mailto:jiangt@muohio.edu).

So in an unordered version of the Canonical Ramsey Theorem, one of the three types of colorings is forced: monochromatic, rainbow, or lexical. Richer [7] further unified the notion of monochromatic and lexical colorings by introducing so-called orderable colorings.

An edge-coloring  $\phi$  of an  $n$ -vertex graph  $G$  is called *orderable* on  $G$  if  $V(G)$  can be ordered as  $x_1, x_2, \dots, x_n$  such that for every pair  $i, j \in [n]$  with  $i < j$  we have  $\phi(x_i, x_j) = c_i$ , where  $c_1, c_2, \dots, c_{n-1}$  is a list of not necessarily distinct colors. Note that if all the  $c_i$ 's are the same then  $\phi$  is monochromatic on  $G$  while if all the  $c_i$ 's are distinct then  $\phi$  is lexical on  $G$ . In general, however, the colors in the list  $c_1, \dots, c_{n-1}$  are arbitrary. So, orderable colorings are much broader than monochromatic colorings and lexical colorings. The Canonical Ramsey Theorem now implies that in any edge-coloring of a large complete graph, there must be a large complete subgraph on which the coloring is either orderable or rainbow. Let  $CR(s, t)$  be the smallest  $N$  such that every coloring of  $E(K_N)$  contains either an orderable  $K_s$  or a rainbow  $K_t$ . Richer obtained the following estimates on  $CR(s, t)$ .

**Theorem 1** ([7]). *Let  $s, t$  be positive integers. We have*

$$\left(\frac{t^2 - t}{2} - 1\right)^{s-2} + 1 \leq CR(s, t) \leq 7^{3-s} t^{4s-4}.$$

Richer also mentioned a stronger lower bound (see Theorem 7) as a remark but did not give a formal proof. We will give a detailed proof of this stronger lower bound. We will also sharpen the upper bound given in Theorem 1 (see Theorem 6) to almost match the lower bound in Theorem 7.

There are several other notions that are inspired by the Canonical Ramsey Theorem. The general theme is to study color patterns in edge-coloring of a host graph (typically the complete graph) in which the number of colors is arbitrary but some constraints are imposed on the color distribution. One notion is that of an  $m$ -good coloring. Given a positive integer  $m$ , an edge-coloring  $\phi$  of a graph  $G$  is  $m$ -good if each color appears at most  $m$  times at each vertex. In other words,  $\phi$  is  $m$ -good if the subgraph induced by each color class has maximum degree no more than  $m$ . Note that a 1-good coloring of  $G$  is simply a proper coloring of  $G$ . The notion of  $m$ -good coloring was a key ingredient in [5], though the concept was not formally defined in the paper. Alon et al. [1] later extensively studied properly-colored and rainbow subgraphs in  $m$ -good colorings of complete graphs. Here a subgraph in an edge-colored graph is *properly colored* if no two incident edges in it have the same color.

Another notion inspired by the Canonical Ramsey Theorem is that of a  $t$ -strong coloring. Given an edge-coloring  $\phi$  of a host graph, we define the *color degree* of any vertex  $x$  to be the number of different colors used on its incident edges. Given a positive integer  $t$ , an edge-coloring of a graph  $G$  is  $t$ -strong if each vertex has color degree at least  $t$ . In other words, we require that at least  $t$  different colors appear at each vertex. Note that an  $m$ -good coloring of  $E(K_N)$  is necessarily  $\lceil \frac{N-1}{m} \rceil$ -strong. However, in general,  $t$ -strong colorings can behave very differently from  $m$ -good colorings since the colors need not distribute uniformly. Axenovich, Jiang, and Tuza [2] studied properly colored and rainbow subgraphs in  $t$ -strong colorings of complete graphs. For a given graph  $H$  with  $k$  edges and an integer  $N \geq n(H)$ , let  $f(N, H)$  denote the least  $t$  such that every  $t$ -strong coloring of  $E(K_N)$  contains a properly colored copy of  $H$ . We are interested in the behavior of  $f(N, H)$  when  $H$  is fixed and  $N$  grows. It was shown [2] that if  $e(H) > n(H)$  then  $f(N, H) \geq N/2$ . On the other hand, if  $H$  is acyclic then  $f(N, H) \leq k$ . The huge drop from  $N/2$  (a growing function of  $N$ ) to  $k$  (a fixed constant) when  $H$  goes from barely containing more than one cycle to being acyclic is very surprising and this leads to a natural question: how does  $f(N, H)$  behave when  $H$  is unicyclic? In particular, how does  $f(N, C_k)$  behave? It follows from a general upper bound in [2] that  $f(N, C_k) \leq N/2 + o(N)$ . However, it is believed that  $f(N, C_k)$  should be upper bounded by a constant that depends only on  $k$ . In particular, the authors of [2] posed the following question.

**Question 2** ([2]). *Given a positive integer  $k$ , does there exist a constant  $\lambda_k$ , depending only on  $k$ , such that for all sufficiently large  $N$  every  $\lambda_k$ -strong coloring of  $E(K_N)$  contains a properly colored cycle of exactly length  $k$ ?*

We will answer the question in the affirmative in Section 3, showing that every  $256k$ -strong coloring of  $E(K_N)$  contains a properly colored  $C_k$ . In other words,  $f(N, C_k) \leq 256k$ . On the other hand, it is shown in [2] that  $f(N, C_k) \geq k/2$ . Hence the bound  $256k$  is best possible except for the constant factor. The spirit of the proof is to show that if a  $256k$ -strong coloring is not  $m$ -good for some adequate  $m$  then there should exist some nice lexically colored subgraph. Then we use an extremally chosen lexical subgraph to find the desired properly colored cycle.

The rest of the paper is organized as follows. In Section 2, we give improved bounds on the unordered Canonical Ramsey number  $CR(s, t)$ . In Section 3, we present a solution to Question 2.

## 2. New bounds on $CR(s, t)$

We start with one of the main results in [1], which extends an earlier result of Babai [3].

**Theorem 3** ([1]). *Let  $m, t$  be positive integers where  $t \geq 2$ . There exist positive absolute constants  $c_1$  and  $c_2$  such that*

1. every  $m$ -good coloring of a complete graph of order at least  $c_1 \frac{m^3}{\ln t}$  contains a rainbow copy of  $K_t$ .
2. There exists an  $m$ -good coloring of a complete graph of order at least  $c_2 \frac{m^3}{\ln t}$  that contains no rainbow copy of  $K_t$ .

As a corollary, we have

**Corollary 4.** *There exists a positive absolute constant  $c$  such that for all positive integers  $N \geq \frac{ct^3}{\ln t}$ , every  $\lfloor \frac{N}{ct^3/\ln t} \rfloor$ -good coloring of  $E(K_N)$  contains a rainbow copy of  $K_t$ .*

We now use this result to substantially improve the estimate on  $CR(s, t)$ . We first prove a simple lemma which will also be used in Section 3.

**Lemma 5.** *Let  $s \geq 2$  be a positive integer and  $p, q \geq 1$  positive reals. Let  $N$  be an integer such that  $N \geq p \cdot q^{s-1}$ . If  $\phi$  is a coloring of  $E(K_N)$  that contains no  $\lfloor L/q \rfloor$ -good copy of  $K_L$  for any  $L \geq pq$ , then  $\phi$  contains an orderable copy of  $K_s$ .*

**Proof.** Let  $G = K_N$ . Let  $A_1 = V(G)$ . By our assumption  $\phi$  is not  $\lfloor \frac{N}{q} \rfloor$ -good on  $G$ . So there exists a vertex  $x_1$  in  $G$  and a color  $c_1$  such that there are at least  $\frac{N}{q} \geq p \cdot q^{s-2}$  edges of color  $c_1$  incident to  $x_1$ . Let  $A_2$  denote the set of vertices in  $G[A_1]$  that are joined to  $x_1$  by edges of color  $c_1$ . Then  $|A_2| \geq p \cdot q^{s-2}$ . If  $s = 2$ , we stop. Otherwise, we can apply the same argument to  $\phi$  restricted to  $G[A_2]$ , and find a vertex  $x_2$  in  $G[A_2]$  and a color  $c_2$  such that  $x_2$  is incident to more than  $\frac{|A_2|}{q} \geq p \cdot q^{s-3}$  edges of color  $c_2$  in  $G[A_2]$ . Let  $A_3$  denote the set of vertices in  $G[A_2]$  connected to  $x_2$  by edges of color  $c_2$ . Then  $|A_3| \geq p \cdot q^{s-3}$ . We can continue this process to obtain a sequence  $A_1 \supseteq A_2 \supseteq \dots \supseteq A_{s-1} \supseteq A_s$ , such that for each  $i = 1, \dots, s-1$ ,  $G[A_i]$  contains a vertex  $x_i$  such that all the edges joining  $x_i$  to  $A_{i+1}$  have color  $c_i$ . Furthermore, for each  $i = 1, \dots, s$ ,  $|A_i| \geq p \cdot q^{s-i}$ . Since  $|A_s| \geq p \geq 1$ , we can let  $x_s$  be any vertex in  $A_s$ . Now,  $x_1, x_2, \dots, x_s$  induce a orderable copy of  $K_s$ . ■

**Theorem 6.** *Let  $s, t \geq 2$  be positive integers. Then there exists a positive absolute constant  $c$  such that*

$$CR(s, t) \leq \left( \frac{ct^3}{\ln t} \right)^{s-1}.$$

**Proof.** Let  $c$  be the constant stated in Corollary 4. Let  $q = \frac{ct^3}{\ln t}$ . Let  $N$  be an integer such that  $N \geq \left( \frac{ct^3}{\ln t} \right)^{s-1} = 1 \cdot q^{s-1}$ . Let  $\phi$  be an edge-coloring of  $G = K_N$ . If for some  $L \geq q$ ,  $G$  contains a copy  $H$  of  $K_L$  such that  $\phi$  restricted to it is  $\lfloor L/q \rfloor$ -good, then by Corollary 4,  $H$  contains a rainbow copy of  $K_t$  and we are done. Hence, we may assume that  $G$  contains no  $\lfloor \frac{L}{q} \rfloor$ -good copy of  $K_L$  for any  $L \geq q = 1 \cdot q$ . By Lemma 5,  $\phi$  contains an orderable  $K_s$ . ■

We now show that the bound given in Theorem 6 is not too far from the best possible by giving the following lower bound. This lower bound was mentioned in [7] as a remark with details left out. We give a proof here for completeness.

**Theorem 7.** *Let  $s, t \geq 3$  be positive integers. There exists a positive absolute constant  $c'$  such that*

$$CR(s, t) \geq \left( \frac{c't^3}{\ln t} \right)^{s-2}.$$

**Proof.** By a special case of the second part of Theorem 3, which was first proved by Babai [3], there exists a positive absolute constant  $c'$  such that there exists a proper edge-coloring  $f$  of a complete graph of order  $\lceil \frac{c't^3}{\ln t} \rceil$  that contains no rainbow  $K_t$ . Let  $q = \lceil \frac{c't^3}{\ln t} \rceil$ . For each  $s = 3, 4, \dots$ , we construct a coloring  $f_s$  of  $E(K_{q^s})$  with no orderable  $K_s$  and no rainbow  $K_t$ . We use induction on  $s$ . For the basis case, let  $s = 3$ . Let  $f = f_3$  be the proper coloring of  $E(K_q)$  with no rainbow  $K_t$ , described as above. It is easy to see that since  $f_3$  is proper, it contains no orderable  $K_3$ . For the induction step, suppose  $f_i$  is a coloring of  $K_{q^i}$  with no orderable  $K_i$ , we define  $f_{i+1}$  as follows. Let  $A_1, \dots, A_q$  be disjoint sets of size  $q^i$ . We color edges in the complete graph on each  $A_i$  using  $f_i$ . To color the edges between different  $A_i$ 's, we use  $f$  indirectly as follows. Let us view  $f$  as a coloring of a complete graph on vertices  $x_1, \dots, x_q$ . Suppose  $f$  uses  $r$  colors. We introduce a set  $S$  of  $r$  new colors not used in  $f_i$ , one for each color in  $f$ . For each pair  $u, v$  where  $u \in A_i, v \in A_j$ , we let  $f_{i+1}(uv) = f'(x_i x_j)$ , where  $f'(x_i x_j)$  denotes the color in  $S$  associated with  $f(x_i x_j)$ .

We now argue that  $f_{i+1}$  is a coloring of  $E(K_{q^s})$  with no orderable  $K_{i+1}$  or rainbow  $K_t$ . Consider a rainbow complete subgraph  $H$  in  $f_{i+1}$ . If  $H$  intersects at least two  $A_j$ 's then by our definition of  $f_{i+1}$ ,  $H$  can contain at most one vertex from each  $A_j$  (otherwise  $H$  would not be rainbow). Hence we may view  $H$  as a complete subgraph on a subset of  $\{x_1, x_2, \dots, x_q\}$ . Since the coloring  $f$  on  $\{x_1, \dots, x_q\}$  has no rainbow  $K_t$  we see that  $H$  has order at most  $t-1$ . Next, we assume that  $H$  lies completely inside one  $A_j$ . Then since  $f_{i+1}$  restricted to  $A_j$  is  $f_i$  which by induction hypothesis contains no rainbow  $K_t$ , we conclude that  $H$  has order at most  $t-1$ . We have argued that  $f_{i+1}$  contains no rainbow  $K_t$ .

Next, consider any orderable complete subgraph  $F$  in  $f_{i+1}$ . We argue that  $F$  has order at most  $i$ . If  $F$  contains three vertices  $u, v, w$  from three different  $A_j$ 's, say  $A_1, A_2, A_3$ , then in  $f$ , vertices  $x_1, x_2, x_3$  would induce an orderable  $K_3$ , contradicting  $f$  being proper. Hence  $F$  contains vertices from at most two different  $A_j$ 's, say  $A_1$  and  $A_2$ . Let  $u$  denote the first vertex in the corresponding ordering of  $V(F)$ . Without loss of generality, suppose  $u \in A_1$ . If  $F$  lies inside  $A_1$ , then  $F$  has order at most  $i-1$  by induction hypothesis. So we may assume that  $F$  contains some vertex  $w$  in  $A_2$ . If  $F$  contains a vertex  $v$  inside  $A_1$  other than  $u$  then  $uv$  and  $uw$  have different colors since the colors used on edges between  $A_1$  and  $A_2$  are different from those used inside  $A_1$ . This contradicts  $u$  being the first vertex of  $F$ . So  $F-u$  must lie completely inside  $A_2$ . Now,  $F-u$  has order at most  $i-1$  by induction hypothesis. Hence,  $F$  has order at most  $i$ . This completes the induction step and the proof. ■

To conclude this section, we mention the following simple fact which was used in several earlier papers. We include its proof for completeness. We will use [Proposition 8](#) in Section 3.

**Proposition 8.** *Let  $a, b$  be positive integers. If  $\phi$  is an orderable coloring of  $E(K_t)$ , where  $t \geq (a-1)(b-1)+2$ , then  $\phi$  contains either a monochromatic  $K_{a+1}$  or a lexical  $K_{b+1}$ .*

**Proof.** Let  $x_1, \dots, x_t$  be an ordering of  $K_t$  such that  $c(x_i, x_j) = c_i$ , where  $i < j$ , and  $c_1, \dots, c_{t-1}$  is a list of not necessarily distinct colors. If some value appears  $a$  times in the list  $c_1, \dots, c_{t-1}$  then we can find a monochromatic  $K_{a+1}$ . Otherwise there must be at least  $b$  different values in the list and we can find a lexical  $K_{b+1}$ . ■

### 3. Properly colored cycles

In this section, we answer [Question 2](#) in the affirmative. We show that for each fixed positive integer  $k$  and a large enough positive integer  $N$ , every  $256k$ -strong coloring of  $E(K_N)$  contains a properly colored cycle of length exactly  $k$ .

**Lemma 9.** *Let  $N, k$  be positive integers where  $N \geq 3k$ . If  $\phi$  is a  $(\frac{N}{3})$ -good coloring of  $E(K_N)$  then  $\phi$  contains a properly colored  $C_k$ .*

**Proof.** Let  $x = x_1$  be any vertex. Starting at  $x_1$  we can easily grow a properly colored path  $P = x_1x_2 \dots x_{k-1}$  of length  $k-2$ . This is because for each  $i$  at step  $i$  we just need to extend the path via an edge  $x_i x_{i+1}$  of a color other than  $\phi(x_{i-1}x_i)$ . Since  $c$  is  $\frac{N}{3}$ -good, this rules out at most  $\frac{N}{3}$  edges joining  $x_i$  to vertices outside  $P$ . Since  $N > \frac{N}{3} + (k-1)$ , we can find a vertex outside the current  $P$  to play the role of  $x_{i+1}$ . Hence the desired path  $P$  exists.

Now, we need to find a vertex outside  $P$  which can complete a properly colored  $C_k$  with  $P$ . Let  $\alpha = c(x_1x_2)$  and  $\beta = c(x_{k-2}x_{k-1})$ . Let  $A$  be the set of vertices that are joined to  $x_1$  by edges of color  $\alpha$  and  $B$  the set of vertices that are joined to  $x_{k-1}$  by edges of color  $\beta$ . Then  $|A|, |B| \leq \frac{N}{3}$ . Since  $N - \frac{N}{3} - \frac{N}{3} - (k-1) \geq 1$ , there is at least one vertex in  $V(K_N) - (V(P) \cup A \cup B)$ . Such a vertex completes a properly colored  $C_k$  with  $P$ . ■

**Lemma 10.** *Let  $k, a, b, N$  be positive integers, where  $N \geq k \cdot (3)^{ab-1}$ . Every coloring  $\phi$  of  $E(K_N)$  contains either a properly colored  $C_k$ , a monochromatic  $K_{a+1}$  or a lexical  $K_{b+1}$ .*

**Proof.** Let  $\phi$  be an edge-coloring of  $G = K_N$ . By [Lemma 5](#),  $\phi$  contains either an  $\frac{1}{3}$ -good copy  $H$  of  $K_L$  for some  $L \geq 3k$  or there exists an orderable  $K_{ab}$ . In the former case, we can apply [Lemma 9](#) to obtain a properly colored  $C_k$ . In the latter case, we may assume that  $ab > 1$ , and since  $ab \geq (a-1)(b-1)+2$ , we can apply [Proposition 8](#). ■

Recall that a *tournament* is a simple directed graph whose underlying graph is a complete graph. The *order* of a tournament is the number of vertices in it. A tournament of order  $p$  is *transitive* if its vertices can be ordered as  $x_1, x_2, \dots, x_p$  such that for each pair  $x_i, x_j$  where  $i < j$ , the arc between  $x_i$  and  $x_j$  is oriented from  $x_i$  to  $x_j$ . It is well known [6] that every tournament of order  $n$  contains a transitive subtournament of order at least  $\log_2 n$ . We will use this fact in the proof of the next lemma.

Given positive integers  $p, t$ , let  $S_{p,t}$  denote the spider with  $p$  legs, each of length  $t$ . That is,  $S_{p,t}$  is the graph obtained from the star  $K_{1,p}$  by subdividing each edge  $t-1$  times.

**Lemma 11.** *Given a  $256k$ -strong edge-coloring  $\phi$  of  $G = K_N$ , where  $N \geq 3^{1785} \cdot k$ , there must exist either a lexically colored  $K_8$  or a properly colored  $C_k$ .*

**Proof.** First assume that  $k$  is odd. Suppose  $k = 2t + 1$ . Let  $u$  be a vertex in  $G$ . Since  $\phi$  is  $256k$ -strong, it is easy to see that there exists a rainbow copy  $F$  of  $S_{256,t}$  centered at  $u$ . Let  $A$  denote the set of leaves of  $F$ . For each  $x \in A$ , let  $e_x$  denote the edge of  $F$  incident to  $x$ . If there exist  $x, y \in A$  such that  $\phi(xy) \notin \{\phi(e_x), \phi(e_y)\}$  then the two legs of  $F$  from  $u$  to  $x$  and  $y$  together with the edge  $xy$  form a properly colored  $C_k$  and we are done. Hence, we may assume that for each pair  $x, y \in A$ ,  $\phi(xy) = \phi(e_x)$  or  $\phi(xy) = \phi(e_y)$ . We will orient the edge  $xy$  from  $x$  to  $y$  if  $\phi(xy) = \phi(e_x)$ , and from  $y$  to  $x$  if  $\phi(xy) = \phi(e_y)$ . Doing this for all pairs  $x, y \in A$  defines a tournament  $D$  on  $A$ . Since  $D$  has order at least  $256 = 2^8$ , by our earlier remark,  $D$  contains a transitive subtournament  $T$  of order  $\log_2 256 = 8$ . Suppose the vertices of  $T$  are  $x_1, \dots, x_8$  such that for all  $i, j \in [8]$  and  $i < j$ ,  $x_i x_j$  is oriented from  $x_i$  to  $x_j$ . By our definition of  $D$ ,  $\phi(x_i x_j) = \phi(e_{x_i})$  for all  $i, j \in [8]$  with  $i < j$ . Since  $\phi(e_{x_1}), \dots, \phi(e_{x_7})$  are distinct, we see that  $\phi$  restricted to  $T$  forms a lexical coloring of  $T$ . So  $T$  is a lexically colored  $K_8$ .

Next we assume that  $k$  is even. Suppose  $k = 2m + 2$ . Since  $N \geq 3^{1785}k = k \cdot 3^{255 \cdot 7}$ , by [Lemma 10](#), there exists either a monochromatic  $K_{256}$ , a lexical  $K_8$ , or a properly colored  $C_k$ . We are done in the latter two cases. Hence we may assume that there is a monochromatic copy  $L$  of  $K_{256}$ . Suppose all the edges in  $L$  have color 1. Let  $A = V(L)$ . Since  $\phi$  has minimum color degree at least  $256k > 256m + 256(m+1)$ , it is easy to see that we can find a collection  $F$  of 256 vertex-disjoint paths of length  $m$  each having one endpoint in  $A$  such that all the edges in this collection have different colors and none uses color 1. Let  $B$  denote the set of the other endpoints of these paths. For each  $x \in B$ , let  $e_x$  denote the edge in  $F$  that is incident to  $x$ . If there are  $x, y \in B$  such that  $\phi(xy) \notin \{\phi(e_x), \phi(e_y)\}$  then the two paths  $P$  and  $Q$  in  $F$  ending at  $x$  and  $y$  together with the edge  $xy$  and the edge in  $L$  between the other two endpoints of  $P$  and  $Q$  form a properly colored  $C_k$  and we are done. Hence, we may assume that for all  $x, y \in B$ ,  $\phi(xy) = \phi(e_x)$  or  $\phi(xy) = \phi(e_y)$ . As in the odd  $k$  case, we define a tournament  $D$  on  $B$  similarly. Since  $D$  has order 256, it has a transitive subtournament of order 8, which yields a lexically colored  $K_8$ . ■

Before we prove our main result, we need a notion that is more restrictive than orderable coloring but is more relaxed than lexical coloring. Given a positive integer  $p$ , an edge-coloring  $\phi$  of  $G = K_N$  is said to be  $p$ -semi-lexical if the vertices of  $G$  can be ordered as  $x_1, \dots, x_N$  such that for all  $i, j \in [N]$  with  $i < j$  we have  $\phi(x_i x_j) = c_i$ , where  $c_1, \dots, c_{N-1}$  is a list of colors in which every  $p$  consecutive ones are distinct. Given a path  $Q$  and two vertices  $x, y$  on  $Q$ , we let  $Q[x, y]$  denote the portion of  $Q$  between  $x$  and  $y$ . We now prove our main result below. Note that the special case  $k = 4$  was solved in [2] as Theorem 4.10 (in which it was shown that every 3-strong edge-coloring of  $E(K_N)$  contains a properly colored  $C_4$ ).

**Theorem 12.** *Let  $k \geq 4$  be an integer. Let  $\phi$  be a  $256k$ -strong edge-coloring of  $G = K_N$ , where  $N \geq 3^{1785} \cdot k$ . Then  $\phi$  contains a properly colored  $C_k$ .*

**Proof.** We may assume that  $k \geq 5$ . We suppose that  $\phi$  does not contain a properly colored  $C_k$  and derive a contradiction. By Lemma 11,  $\phi$  contains either a lexical  $K_8$  or a properly colored  $C_k$ . In the latter case we are done. Hence we may assume the former. In particular,  $\phi$  contains at least one 3-semi-lexical clique. Let  $L$  be a largest 3-semi-lexical clique in  $G$  (under  $\phi$ ). Let  $p = |V(L)|$ . Then  $p \geq 8$  by our discussion. Let  $x_1, \dots, x_p$  be an ordering of  $V(L)$  such that for all  $i, j \in [p]$ ,  $i < j$ , we have  $\phi(x_i x_j) = c_i$ , where  $c_1, \dots, c_{p-1}$  is a list of colors in which every 3 consecutive ones are distinct. Let  $Q = x_1 x_2 \dots x_p$ . Then  $Q$  is a properly colored spanning path of  $L$ .

**Claim 1.** Let  $P$  be a properly colored path of length  $t$ , where  $1 \leq t \leq k - 3$ , such that  $u = x_1$  is an endpoint of  $P$  and  $V(P) \cap V(Q) = \{x_1\}$ . Let  $v$  denote the other endpoint of  $P$ . Let  $e_u, e_v$  denote the edges of  $P$  incident to  $u$  and  $v$ , respectively. Suppose  $\phi(e_u) \neq c_1$ . Then for all  $s$  satisfying  $k - t + 1 \leq s \leq p$ , we have  $\phi(vx_s) = \phi(e_v)$ . In particular, at most  $k - t + 1$  different colors are used on the edges from  $v$  to  $V(Q)$ .

**Proof of Claim 1.** Suppose otherwise that for some  $s$ ,  $k - t + 1 \leq s \leq p$  we have  $\phi(vx_s) \neq \phi(e_v)$ . Then  $P' = P \cup vx_s$  is a properly colored path of length  $t + 1$  between  $x_1$  and  $x_s$  that is internally disjoint from  $Q$ .

For convenience, let  $\beta = \phi(vx_s)$ . Suppose first that  $c_{k-t-1} \neq \beta$ . Then  $P' \cup Q[x_1, x_{k-t-1}] \cup x_{k-t-1}x_s$  is a properly colored cycle of length  $(t + 1) + (k - t - 1) = k$ , and we obtain a contradiction. So suppose  $c_{k-t-1} = \beta$ . Since  $L$  is 3-semi-lexical,  $c_{k-t-2}, c_{k-t-1}$  and  $c_{k-t}$  are distinct. So, in particular,  $c_{k-t} \neq \beta$  and  $c_{k-t-2} \neq c_{k-t}$ . Now,  $P' \cup Q[x_1, x_{k-t-2}] \cup x_{k-t-2}x_{k-t} \cup x_{k-t}x_s$  is a properly colored cycle of length  $k$  and we obtain a contradiction. This proves the first part of the claim. The second part of the claim follows immediately from the first part. ■

Note that it is possible that  $p$  is so small that  $p < k - t + 1$  and no  $s$  satisfies the condition of Claim 1. In that case, the claim holds vacuously. In particular, it is still the case that at most  $k - t + 1$  colors are used on the edges from  $v$  to  $V(Q)$ . Let  $y_1 = x_1$ . Let  $y_2$  be a neighbor of  $y_1$  outside  $Q$  such that  $\phi(y_1 y_2) \neq c_1$ . Since all the edges from  $y_1$  to  $V(Q) - y_1$  have color  $c_1$  and  $y_1$  has color degree at least  $256k$ ,  $y_2$  clearly exists. By Claim 1, at most  $k - 1 + 1 = k$  different colors are used on the edges from  $y_2$  to  $V(Q)$ . Since  $y_2$  has color degree at least  $256k$ , there must exist a neighbor  $y_3$  of  $y_2$  outside  $V(Q) \cup \{y_1, y_2\}$  such that  $\phi(y_2 y_3) \neq \phi(y_1 y_2)$ . We can continue like this to find a properly colored path  $P = y_1 y_2 \dots y_{k-4}$  of length  $k - 5$  such that  $y_1 = x_1$  and  $V(P) \cap V(Q) = \{x_1\}$ . Let  $w = y_{k-4}$ . By Claim 1, at most  $k - (k - 5) + 1 = 6$  different colors are used on the edges from  $w$  to  $V(Q)$ . Also at most  $k - 5$  colors are used on the edges from  $w$  to  $V(P) - w$ . Since  $w$  has color degree at least  $256k$ , we can very easily find a set  $U = \{u_1, u_2, \dots, u_{256}\}$  of size 256 outside  $P \cup Q$  such that  $\phi(wu_1), \phi(wu_2), \dots, \phi(wu_{256})$  are all different and they are different from the color on the edge of  $P \cup Q$  incident to  $w$  (which is  $y_{k-5}w$  if  $k \geq 6$  or  $x_1 x_2$  if  $k = 5$ ).

**Claim 2.** For each  $i \in [256]$  and  $s$  with  $5 \leq s \leq p$ , we have  $\phi(u_i x_s) = \phi(wu_i)$ . Also, for all  $i, j \in [256]$ , we have  $\phi(u_i u_j) \in \{\phi(wu_i), \phi(wu_j)\}$ .

**Proof of Claim 2.** For each  $i \in [256]$ ,  $P \cup wu_i$  is a properly colored path of length  $k - 4$  that satisfies the conditions of Claim 1. By Claim 1, we have  $\phi(u_i x_s) = \phi(wu_i)$  for all  $s$  satisfying  $5 \leq s \leq p$ . This proves the first part.

Next, suppose for some  $i, j \in [256]$ ,  $\phi(u_i u_j) \notin \{\phi(wu_i), \phi(wu_j)\}$ . By our discussion above,  $\phi(u_i x_p) = \phi(wu_i)$ . On the other hand, observe that  $P \cup wu_j \cup u_j u_i$  is a properly colored path of length  $k - 3$  satisfying the conditions of Claim 1 with  $u_j u_i$  being the edge incident to  $u_i$ . By Claim 1,  $\phi(u_i x_p) = \phi(u_j u_i) \neq \phi(wu_i)$ , a contradiction. Hence no such  $i, j$  exist. ■

Now, as in the proof of Lemma 11, we define a tournament  $D$  on  $U$  by orienting an edge  $u_i u_j$  from  $u_i$  to  $u_j$  if  $\phi(u_i u_j) = \phi(wu_i)$ , and from  $u_j$  to  $u_i$  if  $\phi(u_i u_j) = \phi(wu_j)$ . Since  $D$  has order  $256 = 2^8$ , it contains a transitive subtournament  $T$  of order 8. Without the loss of generality, suppose  $V(T) = \{u_1, u_2, \dots, u_8\}$  and that for all  $i, j \in [8]$ ,  $i < j$ , the edge  $u_i u_j$  is oriented from  $u_i$  to  $u_j$ .

For each  $i \in [8]$ , let  $b_i = \phi(wu_i)$ . By our definition of  $D$  and  $T$ , for each  $i \in [8]$ , all the edges from  $u_i$  to  $u_{i+1}, u_{i+2}, \dots, u_8$ , and to  $x_5, x_6, \dots, x_p$  all have color  $b_i$ . Since  $b_1, \dots, b_8$  are distinct, we can find five of them that are different from  $c_5, c_6, c_7$ . Without loss of generality, suppose  $b_1, b_2, \dots, b_5$  are different from  $c_5, c_6, c_7$ . But now,  $L' = L - \{x_1, x_2, x_3, x_4\} \cup \{u_1, u_2, u_3, u_4, u_5\}$  induces a 3-semi-lexical clique, with associated vertex ordering  $u_1, u_2, \dots, u_5, x_5, x_6, \dots, x_p$ , which is larger than  $L$ , contradicting our choice of  $L$ . This completes our proof. ■

Note that the coefficient of  $k$  used in Theorem 12 was chosen mainly for convenience. It is not hard to improve the coefficient 256. Our main goal was to show that for some absolute constant  $c$  every  $ck$ -strong edge-coloring of  $K_N$ , where  $N$  is sufficiently large, contains a properly colored  $C_k$ . It remains an interesting problem to determine the optimal value of  $c$ .

#### 4. Concluding remarks

1. The idea of using oriented edges to model color conflict was initially suggested to the author by Maria Axenovich in a different project. It turns out to be a useful tool dealing with properly colored graphs and could potentially be applied elsewhere.

2. Our approach in this paper seems quite promising for dealing with  $t$ -strong colorings. Most of the difficulty in dealing with extremal questions concerning  $t$ -strong colorings can be attributed to the potentially highly unbalanced color distribution. However, we have seen here that such highly unbalanced color distribution tends to lead to large lexically colored or semi-lexically colored subgraphs, which can be turned into our advantage.

3. Recall that for a given graph  $H$  and a positive integer  $N$ ,  $f(N, H)$  denote the least  $t$  such that every  $t$ -strong coloring of  $E(K_N)$  contains a properly colored copy of  $H$ . We have now understood the function  $f$  quite well. As mentioned in the introduction, for graphs  $H$  with  $e(H) > n(H)$ , we know  $f(n, H) \geq N/2$  (which is within a factor of 2 from the trivial upper bound  $N - 1$ ). The same is true if  $H$  has a component  $F$  with  $e(F) > n(F)$ . Now, if each component of  $H$  satisfies  $e(F) \leq n(F)$ , then each component is a tree or a unicyclic graph, in which case we should have  $f(N, H) = O(k)$ , where  $k = e(H)$ . This is because if  $F$  is a tree with  $m$  edges then we know  $f(N, F) \leq m$  and if  $F$  is unicyclic graph with  $m$  edges then once we find a properly colored cycle of the desired length we can easily branch off that cycle to find a properly colored copy of  $F$ .

4. For a given graph  $H$  and a positive integer  $N$ , let  $g(N, H)$  denote the least  $t$  such that every  $t$ -strong coloring of  $E(K_N)$  contains a rainbow copy of  $H$ . Our understanding of the function  $g$  is still quite limited. See [2] for several open questions.

#### Acknowledgements

The author thanks the referee for the helpful suggestions and Maria Axenovich, Penny Haxell, Dhruv Mubayi, and Oleg Pikhurko for their helpful comments in various occasions.

Research partially supported by the National Security Agency under grant number H98230-07-1-027.

#### References

- [1] N. Alon, T. Jiang, Z. Miller, D. Pritikin, Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints, *Random Structures and Algorithms* 23 (2003) 409–433.
- [2] M. Axenovich, T. Jiang, Zs. Tuza, Local anti-Ramsey numbers of graphs, *Combinatorics, Probability, and Computing* 12 (2003) 495–511.
- [3] L. Babai, An anti-Ramsey theorem, *Graphs and Combinatorics* 1 (1985) 23–28.
- [4] P. Erdős, R. Rado, A combinatorial theorem, *Journal of the London Mathematical Society* 25 (4) (1950) 249–255.
- [5] H. Lefmann, V. Rödl, On Erdős–Rado numbers, *Combinatorica* 15 (1995) 85–104.
- [6] J.W. Moon, *Topics on Tournaments*, Holt, Rinehart and Winston, New York-Montreal, Quebec-London, 1968.
- [7] D. Richer, Unordered canonical Ramsey numbers, *Journal of Combinatorial Theory. Series B* 80 (2000) 172–177.
- [8] D.B. West, *Introduction to Graph Theory*, second ed., Prentice Hall, New Jersey, 2001.