Constrained Ramsey Numbers of Graphs

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Abstract: Given two graphs G and H, let $f(G,H)$ denote the minimum integer n such that in every coloring of the edges of K_n, there is either a copy of G with all edges having the same color or a copy of H with all edges having different colors. We show that $f(G,H)$ is finite iff G is a star or H is acyclic. If S and T are trees with s and t edges, respectively, we show that $1+s(t-2)/2 \leq f(S,T) \leq (s-1)(t^2+3t)$. Using constructions from design theory, we establish the exact values, lying near $(s-1)(t-1)$, for $f(S,T)$ when S and T are certain paths or star-like trees. J Graph Theory 42: 1–16, 2003

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1. INTRODUCTION

A subgraph of an edge-colored graph is *monochromatic*, iff all of its edges have the same color and *rainbow* iff all of its edges have different colors. In classical Ramsey theory, we fix k and seek the minimum n such that every k-coloring of the edges of K_n forces a monochromatic copy of a given graph G. In anti-Ramsey theory, we fix n, and ask for the minimum k such that every k-coloring of the edges of K_n yields a rainbow copy of a given graph G. If we fix neither n nor k, we can still ask what types of subgraphs are forced in every coloring of the edges of complete graphs. Gyárfás et al. [15] introduced a *local* version of classical Ramsey theory by changing the constraint on the number of colors used from a *global* constraint to a *local* constraint. Gyárfás et al. [15] defined a *local* k-coloring of a graph to be an edge-coloring in which at most k different colors appear on the edges incident to each vertex. Fixing an integer k and a graph G, they defined the *local k-Ramsey number* of G, denoted by $R^k_{loc}(G)$, to be the minimum n such that every local k-coloring of K_n contains a monochromatic copy of G. Since every k-coloring is a local k-coloring, we have $R^k_{loc}(G) \geq R^k(G)$ always, where $R^k(G)$ is the classical k-color Ramsey number of G. Truszczyński and Tuza [25] proved for connected graphs G that $R^k_{loc}(G) \leq c_k \cdot R^k(G)$, where c_k is a constant depending only on k. Various work on local k-Ramsey numbers can be found in [6,7,14,15,21,25].

In this paper, we investigate a further generalization of local k-Ramsey theory by playing off the existence of a monochromatic copy of G against the existence of a rainbow copy of H. Specifically, given any two graphs G and H, let $f(G,H)$ denote the minimum integer n such that every edge-coloring of K_n contains either a monochromatic copy of G or a rainbow copy of H. Note that $R^k_{loc}(G)$ is exactly $f(G, K_{1,k+1})$, as a local k-coloring is exactly a coloring that avoids rainbow copies of the star $K_{1,k+1}$. We obtain an upper bound on $f(G,H)$ in terms of local Ramsey numbers of G when H is a tree in Section 2, and we consider $f(G, K_{1,k+1})$ in Section 5. Our results extend some of the ones obtained in [15].

In the study of classical Ramsey numbers $R^k(G)$, the overall number of colors is fixed. In the study of local Ramsey numbers $R^k_{loc}(G)$, a constraint on the number of colors used is given locally. In the study of $f(G,H)$, no constraint is placed on the number of colors, but the distribution of the colors is constrained so as to avoid rainbow copies of H. In all three cases, we want to avoid monochromatic copies of G. Thus, if we fix H and let G vary, we can view $f(G,H)$ as a *constrained Ramsey number* of G. Similarly, if we fix G and let H vary, we may regard $f(G,H)$ as a *constrained anti-Ramsey number* of H.

The finiteness of $f(G,H)$ when either G is a star or H is acyclic follows readily from the classical *Canonical Ramsey Theorem* of Erdős and Rado [12] which asserts that *every* edge-coloring of a sufficiently large K_n contains large complete portions on which the coloring is of one of several canonical types. Two of these are monochromatic and rainbow. The others may be described as follows. An edge-coloring of a graph G is *lexical* provided there is a total order of the vertices of G so that two edges uv and xy have the same color iff they have the same smaller endpoint, i.e., $\min(u, v) = \min(x, y)$. Using this notion, we now give the Erdős–Rado theorem in a slightly different (and weaker) version from its original statement.

Theorem 1.1 (Erdős–Rado [12]). *Given a positive integer m, there exists an integer n such that in every edge-coloring c of K_n there is a complete subgraph of order m on which the restriction of c is either monochromatic, rainbow, or lexical.*

In this paper, we consider only graphs containing no isolated vertices, and we will use the following notation. If G is a graph, then $V(G), E(G), n(G)$, and $e(G)$ denote the vertex set and the edge set of G and their cardinalities, respectively. If $U \subseteq V(G)$, then $G[U]$ denotes the subgraph of G induced by U. Given an edge-coloring c of G, the *color-degree* $d_G^c(v)$ of a vertex v in G is the number of colors appearing on the edges incident with v in G. Also, $ex(n, G)$ denotes the *Turán number* of G, namely, the maximum number of edges in a graph on n vertices which does not contain G as a subgraph. We denote the path on n vertices by P_n and the star with t edges by $K_{1,t}$. The set of integers $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

2. EXISTENCE AND GENERAL BOUNDS

Proposition 2.1. *Given graphs G and $H, f(G, H)$ exists if and only if G is a star or H is acyclic.*

Proof. In any lexical edge-coloring of K_n, every color class is a star, so if G is not a star, it cannot appear in a single color class. Moreover, in any cycle, the two edges at the smallest vertex receive the same color. Thus if H contains a cycle, a rainbow copy of H cannot occur in any lexical coloring. Thus if G is not a star and H contains a cycle, lexical edge-colorings of K_n fail to contain either a monochromatic G or a rainbow H for all n. Thus $f(G, H)$ fails to exist in this case.

Next, suppose that either G is a star or H is acyclic. Let $m = \max\{n(G), n(H)\}$, and choose the n guaranteed by the Erdős–Rado Canonical Ramsey Theorem for this m. Let c be an arbitrary coloring of $E(K_n)$. By the Erdős–Rado Canonical Ramsey Theorem, there exists a subset S of order m such that c restricted to S is either monochromatic, rainbow, or lexical. If c is monochromatic on S, then S (and hence K_n) contains a monochromatic copy of G, since $n(G) \leq m$. Similarly, if c is rainbow on S, then there is a rainbow copy of H, since $n(H) \leq m$.

It remains to consider the case where c is lexical on S. If G is a star, then mapping its center to the smallest vertex of S leads to a monochromatic embedding of G in S. Hence, suppose H is acyclic. Arbitrarily root each component of H, and consider the partial order on the nodes of H given by $x \preceq y$, iff x is a descendant of y (in the same rooted component). Then the minimum over any edge is the child. Since no node is a child of more than one parent, all edges have different minima. Thus taking any linear extension of $\preceq$ and using it to define an order-preserving injection of $V(H)$ into S will result in a rainbow embedding of H into K_n. ■

Next, we derive general lower bounds on $f(G,H)$. First, we make an easy observation.

Proposition 2.2. *Let G,H be two graphs, where H has k edges. Then $f(G,H) \geq R^{k-1}(G)$.*

Proof. Let $n = R^{k-1}(G) - 1$. By definition, there exists a $(k-1)$-coloring of $E(K_n)$ that has no monochromatic copy of G. Since such a coloring uses at most $k-1$ colors, it also contains no rainbow copy of H. Hence $f(G,H) \geq n + 1 = R^{k-1}(G)$. ■

In the next two theorems, we give some general lower bounds on $f(G,H)$ by obtaining lower bounds on $R^{k-1}(G)$.

Theorem 2.1. *Let G be a graph with maximum degree $\Delta(G)$ and with p vertices in its largest component. Let H be a graph with k edges. Then $f(G,H) \geq R^{k-1}(G) \geq rm + 1$, where $r = \max\{\lfloor \frac{p-1}{2} \rfloor, \Delta(G) - 1\}$ and $m = k-1$ if k is odd and $m = k-2$ if k is even.*

Proof. Let $n = rm$, and note that m is always even. It suffices to define a $(k-1)$-coloring of $E(K_n)$ containing no monochromatic copy of G. Partition the vertices of K_n into m subsets of order r, and denote them by $A_1, \ldots, A_m$, respectively. Assign the same color, call it α, to all the edges within the A_i's. Next, color the edges between different A_i's using new colors as follows. Construct a complete graph K on m vertices $x_1, \ldots, x_m$. Let c be a proper coloring of $E(K)$ using $m-1$ colors $1, \ldots, m-1$; such a coloring exists since a complete graph of even order has a 1-factorization. Now use this coloring c of $E(K)$ to color the edges between different A_i's by assigning color $c(x_i x_j)$ to all the edges between A_i and A_j.

In this coloring, a component in the subgraph induced by any color class has order at most $2r$ and maximum degree at most r. If $r = \lfloor \frac{p-1}{2} \rfloor$, then $2r \leq p-1$, so the largest component of G will not fit into a single color class. If $r = \Delta(G) - 1$, then the vertex of largest degree in G will not fit into a single color class. Hence in either event, there is no monochromatic copy of G in this coloring. Overall, $(m-1) + 1 \leq k-1$ colors are used, so it is a $(k-1)$-coloring. ■

In an edge decomposition of K_n into K_m's, a set of K_m's in the decomposition that together partitions the vertex set of K_n is called a *parallel class*. An edge decomposition of K_n into K_m's is *resolvable*, iff the K_m's can be partitioned into parallel classes. Note that if K_n admits an edge decomposition into K_m's (the blocks of a 2-design), then

(1) there are $n(n-1)/m(m-1)$ blocks K_m;
(2) there are $(n-1)/(m-1)$ blocks K_m at each vertex of K_n;
(3) if the decomposition is resolvable, there are n/m blocks K_m in a parallel class;
(4) if the decomposition is resolvable, there are $(n-1)/(m-1)$ parallel classes.

Since the above counts must be integral, they lead to standard divisibility (and hence congruence) conditions on the parameters n and m. These are the standard necessary conditions for a decomposition to exist. In certain cases, decompositions can be used as in Theorem 2.1 to construct colorings which provide bounds on constrained Ramsey numbers. Although, the lower bound obtained in Theorem 2.1 is sharp in some cases (for instance when G and H are both stars and q is odd, see Section 4), when q is large enough and belongs to the right congruence class modulo $p-1$, we can strengthen the lower bound.

Theorem 2.2. *Let G be a connected graph with p vertices and H a graph with k edges. If k is sufficiently large and* $k \equiv 2 \pmod{p-1}$, *then* $f(G,H) \geq R^{k-1}(G) \geq (p-2)(k-1)+2$.

Proof. Let $n=(p-2)(k-1)+1$. It suffices to define a $(k-1)$-coloring that contains no monochromatic copy of G. According to [1], when k is sufficiently large and $k \equiv 2 \pmod{p-1}$, there exists a resolvable edge decomposition of K_n into K_{p-1}'s. Such a resolvable decomposition contains $(n-1)/(p-2) = k-1$ parallel classes. By assigning color i to the ith parallel class for all $i \in [k-1]$, we obtain a coloring of the edges of K_n using $k-1$ colors. In each color class, the largest component has order $p-1$ and hence contains no copy of G. ■

Next, we derive an upper bound on $f(G,H)$ in terms of local Ramsey numbers of G when H is a tree. Given an edge-coloring c of a graph G, we define the *color-degree* of a vertex v in G, to be denoted by $d_G^c(v)$, as the number of colors appearing on the edges incident with v in G. Given graphs G and H and a coloring of $E(G)$, a *rainbow embedding* of H in G is an embedding, σ, of H in G whose image $\sigma(H)$ is rainbow.

Lemma 2.1. *Let G be graph and c a coloring of* $E(G)$. *Let T be a tree with m edges, v a leaf of T, and u the neighbor of v in T. Suppose there is a rainbow embedding σ of* $T-v$ *in G satisfying* $d_G^c(\sigma(u)) \geq 2m-2$, *then σ can be extended to a rainbow embedding of T in G.*

Proof. Let A denote the set of edges in G that $\sigma(u)$ sends to vertices in $\sigma(T-v)$; clearly $|A| \leq m-1$. Let B denote the set of edges in $\sigma(T-v)$; we have

$|B| = m-1$. Note that $|A\cap B| \geq 1$. Since $d_G^c(\sigma(u)) \geq 2m-2 > |A\cup B|$, there exists an edge e incident to $\sigma(u)$ in G whose color is not used on any edge in $|A\cup B|$. In particular, the other endpoint of e does not lie in $\sigma(T-v)$. Extending $\sigma(T-v)$ through e yields a rainbow copy of T in G. ■

Lemma 2.2. *Let G be a graph and c a coloring $E(G)$. Let T be a tree with $t \geq 2$ edges, and T' be the subtree of T obtained by deleting all leaves of T. Suppose $G[U]$ contains a rainbow copy of T', where $U = \{v \in V(G) : d_G^c(v) \geq 2t-2\}$. Then G contains a rainbow copy of T.*

Proof. Suppose the leaves of T are $v_1, \ldots, v_l$. Let $T_0 = T$ and for $i \in [l]$, let $T_i = T_{i-1} - v_i$. Note that $T_l = T'$. Let σ be a rainbow embedding of T' in $G[U]$, whose existence is assumed as a condition. Applying Lemma 2.1 successively to $T' = T_l, T_{l-1}, \ldots, T_1, T_0 = T$ allows us to extend σ to a rainbow embedding of T in G in l steps (note that conditions of Lemma 2.1 are satisfied after each step). ■

We now use the above lemma to recursively derive a general bound on the constrained Ramsey number $f(G,T)$, when T is a tree, in terms of local Ramsey numbers of G. As noted in the introduction, when $T = K_{1,t}$, we have $f(G,T) = R_{loc}^{t-1}(G)$ by definition.

Lemma 2.3. *Let T be a tree with $t \geq 2$ edges which is not a star, and let T' be the subtree of T obtained by deleting all leaves of T. For any graph G, we have*

$$f(G,T) \leq f(G,T') + R_{loc}^{2t-3}(G) - 1.$$

Proof. Let $n = f(G,H)$, and let c be a coloring of $E(K_{n-1})$ containing no monochromatic copy of G or rainbow copy of T. For convenience, write $K = K_{n-1}$. Let $U = \{v \in V(G) : d_G^c(v) \geq 2t-2\}$, and set $W = V(K) - U$. Consider c restricted to $K[U]$ (which is a complete graph of order $|U|$). It contains no monochromatic copy of G, and in addition, by Lemma 2.2, it has no rainbow copy of T'. Hence we have $|U| \leq f(G,T') - 1$.

On the other hand, by the definition of W, the restriction of c to $K[W]$ (which is a complete graph of order $|W|$) is a local $(2t-3)$-coloring of $K[W]$. Moreover, it has no monochromatic copy of G by choice of c. Hence, we have $|W| \leq R_{loc}^{2t-3}(G) - 1$. So, we have $n-1 = |U| + |W| \leq f(G,T') - 1 + R_{loc}^{2t-3}(G) - 1$, whence $f(G,T) = n \leq f(G,T') + R_{loc}^{2t-3}(G) - 1$ as desired. ■

Given a tree T with t edges and radius r, let $T_1, T_1, \ldots, T_r$ be subtrees of T such that $T_1 = T$ and, for $i \geq 2$, T_i is obtained from T_{i-1} by deleting all leaves of T_i. Notice that T_r is a star if T is unicentral and a single edge (and hence a star) if T is bicentral.

By applying Lemma 2.3 successively and using the fact that $f(G,T_r) = R_{loc}^{t_r-1}(G)$ (since T_r is a star), we obtain the following bound.

Theorem 2.3. *Let T be a tree with $t \geq 2$ edges and radius r, and let $T_1,\dots,T_r$ be defined as above with $e(T_i) = t_i$. For any graph G, we have*

$$f(G,T) \leq \sum_{i=1}^{r-1} R_{loc}^{2t_i-3}(G) + R_{loc}^{t_r-1}(G) - r + 1.$$

Corollary 2.1. *Let T be a tree with $t \geq 2$ edges and radius r, and let G be any graph. We have $f(G,T) = R_{loc}^{t-1}(G)$ if $r = 1$, and $f(G,T) \leq r \cdot R_{loc}^{2t-3}\ (G)$ otherwise.*

Corollary 2.2. *For every integer $t \geq 2$, there is a constant b_t such that for every connected graph G and every tree T with t edges, $f(G,T) \leq b_t \cdot R^{2t-3}(G)$.*

Proof. As noted in the Introduction, Truszczyński and Tuza [25] established the existence, for each integer k, of a constant c_k such that $R_{loc}^{k}(G) \leq c_k \cdot R^k(G)$ holds for every connected graph G. Since the radius of a tree with t edges is at most $\lceil t/2 \rceil$, if we take $b_t = \lceil t/2 \rceil \cdot c_{2t-3}$, then the desired result follows from Corollary 2.1. ■

3. CONSTRAINED RAMSEY NUMBERS FOR A PAIR OF TREES

From now on, unless otherwise specified, we consider $f(S,T)$ where S is a tree with s edges and T is a tree with t edges. We will first use some facts about Turán numbers of trees to obtain bounds on the local Ramsey numbers of S. Then we apply Theorem 2.3 to establish a general upper bound on $f(S,T)$.

Recall that given an integer n and a graph G, the *Turán number* of G, denoted by $ex(n,G)$, is defined as the maximum number of edges in a graph on n vertices which does not contain G as a subgraph. It is well known that every graph on n vertices with more than $n(d-1)$ edges contains a subgraph of minimum degree at least d and that every graph with minimum degree at least d contains every tree on d edges as a subgraph. These immediately imply the following well-known bound on the Turán number of a tree.

Proposition 3.1. *If S is a tree with s edges, then $ex(n,S) \leq n(s-1)$.*

In 1964, Erdős and Sós [9] made the following well-known conjecture strengthening the above bound.

Conjecture 3.1 ([9]). *If S is a tree with s edges, then $ex(n,S) \leq n(s-1)/2$.*

Remark 3.1. An approximate version of this conjecture has been verified by Ajtai, Komlós, and Szemerédi (see [22]). The exact version of the conjecture has been verified for some special families of trees (see [27]), including trees with s edges which have a vertex adjacent to at least $(1/2)(s-1)$ leaves caterpillars and spiders of diameter at most 4. A tree is a *caterpillar* if deleting its leaves yields a path. A *spider* (also called an *aster*) is obtained by subdividing edges of a star.

In the sequel, we will make use of a parameter α_S defined as follows for a tree S with s edges:

$$\alpha_S = sup\left\{\frac{ex(n,S)}{n(s-1)/2} : n \geq 1\right\}$$

Proposition 3.1 asserts that α_S exists and is at most 2. Disjoint union of K_s's (for n divisible by s) shows that $\alpha_S \geq 1$. The Erdős–Sós [9] Conjecture says that equality holds for all trees S.

Proposition 3.2. *Let S be a tree, and let $\alpha_S = sup\{\frac{ex(n,S)}{n(s-1)/2} : n \geq 1\}$.* We have $1 \leq \alpha_S \leq 2$. *In addition, if the Erdős–Sós conjecture holds for S, then $\alpha_S = 1$.*

A graph H which realizes the maximum in the definition of the Turán number $ex(n,G)$ is said to be *G-extremal.* Here, we will make use of a stronger notion. For a tree S, a graph H which realizes the supremum in the definition of α_S will be called *strongly S-extremal.* That is, H is strongly S-extremal iff

$$e(H) = ex(n(H),S) = \alpha_S n(s-1)/2.$$

The argument of the previous paragraph shows that K_s is strongly S-extremal if S satisfies the Erdős–Sós Conjecture.

The following bound on local Ramsey numbers will also be used here and again in Section 5. A similar result was obtained independently in [6].

Theorem 3.1. *For any tree S, we have $R^k_{loc}(S) \leq \alpha_S(s-1)k+2$. Moreover, equality holds if and only if there exists a decomposition of $E(K_{\alpha_S(s-1)k+1})$ into subgraphs F_j such that each F_j is connected and strongly S-extremal and each vertex of $K_{\alpha_S(s-1)k+1}$ is contained in exactly k of the F_j.*

Proof. Let $n = R^k_{loc}(S) - 1$, and let c be a local k-coloring of $E(K_n)$ that does not contain a monochromatic copy of S. Suppose c uses colors $1,\ldots,p$. For each $i \in [p]$, let H_i denote the subgraph (with no isolated vertices) induced by the edges with color i. Since c contains no monochromatic S, it follows that H_i does not contain S as a subgraph. By the definition of α_S, we have $e(H_i) \leq \alpha_S \cdot n(H_i)(s-1)/2$. Hence, we obtain

$$\sum_{i=1}^{p} e(H_i) \leq [\alpha_S(s-1)/2]\sum_{i=1}^{p} n(H_i).$$

Since the H_i's decompose K_n, we have $\Sigma_i e(H_i) = n(n-1)/2$. On the other hand, since each vertex in K_n is contained in at most k of the H_i's, we have $\Sigma_i n(H_i) \leq$

nk. This yields $n(n-1)/2 \leq [\alpha_S(s-1)/2]nk$, which gives $n-1 \leq \alpha_S(s-1)k$. Therefore, $R^k_{loc}(S) = n+1 \leq \alpha_S(s-1)k+2$. Equality holds iff equality holds throughout the above argument. That is, each H_i must satisfy $e(H_i) = \alpha_S n(H_i)(s-1)/2 = ex(n(H_i), S)$, and each vertex in K_n must be contained in exactly k of the H_i's. The first condition says each H_i is strongly S-extremal, and the second is the desired local constraint. Now, by taking the components of the H_i's to be the F_j's, we establish the second statement of the theorem. ■

Since as noted above, $1 \leq \alpha_S \leq 2$ for any tree, with the lower equality holding for trees satisfying the Erdős-Sós conjecture, the following corollary is immediate.

Corollary 3.1. *If S is a tree with s edges, then $R^k_{loc}(S) \leq 2(s-1)k+2$. Moreover, if the Erdős–Sós conjecture holds for S, then $R^k_{loc}(S) \leq (s-1)k+2$.*

We will see in Section 5 that $R^k_{loc}(S) = (s-1)k+2$ holds for infinitely many trees S. Now we apply Theorem 2.3 to derive a general upper bound on $f(S,T)$ on the order of $s \cdot t^2$, where S, T are trees with s and t edges, respectively. To simplify the derivation, a couple of lemmas are useful. The next one can be easily verified, we omit the proof.

Lemma 3.1. *Let $m_1 \geq m_2 \geq \cdots \geq m_r$ be a nonincreasing sequence of r positive integers. Let $\phi = m_1 + \cdots + m_{i-1}$ and $\theta = m_i + \cdots + m_r$ for a fixed but arbitrary index i. Then*

$$\theta \leq \left(1 - \frac{i-1}{r}\right)(\theta + \phi).$$

Let T be a tree and T' a tree obtained from T by deleting all leaves of T. Observe that T' has at most as many leaves as T.

Lemma 3.2. *Let T be a tree with t edges and radius r. Let $T_1, T_2, \ldots, T_r$ be subtrees of T such that $T_1 = T$ and for $i \geq 2$, T_i is obtained from T_{i-1} by deleting all leaves of T_{i-1}. If T_i has t_i edges, then $\Sigma^r_{i=1} t_i \leq t(r+1)/2$.*

Proof. For $i = 1, 2, \ldots, r$, let m_i denote the number of leaves in T_i. By the observation preceding the lemma, the sequence m_i is nonincreasing. Now m_i is also the number of pendant edges of T_i and hence the number of edges discarded from T_i in forming T_{i+1}. It follows that the number t_i of edges of T_i is just $m_i + m_{i+1} + \cdots + m_r$. Applying Lemma 3.1, we see that $t_i \leq t(1 - \frac{i-1}{r})$. Summing we obtain

$$\sum_{i=1}^{r} t_i \leq \sum_{i=1}^{r} t\left(1 - \frac{i-1}{r}\right) = \sum_{j=1}^{r} t\left(\frac{j}{r}\right) = t(r+1)/2$$

as claimed. ■

Theorem 3.2. *Let S be a tree with s edges, and let T be a tree with* $t \geq 2$ *edges and radius r. If* $r = 1$, *then* $f(S,T) \leq \alpha_S(s-1)(t-1)+2$. *Otherwise, we have* $f(S,T) \leq \alpha_S(s-1)t(r+1) \leq 2(s-1)t(r+1)$.

Proof. If $r = 1$, then $T = K_{1,t}$, and $f(S,T) = R_{loc}^{t-1}(S)$, so the assertion follows from Corollary 3.1. So we assume $r \geq 2$, and let T_i and t_i be as in Lemma 3.2. We now have

$$\begin{aligned}
f(S,T) &\leq \sum_{i=1}^{r-1} R_{loc}^{2t_i-3}(S) + R_{loc}^{t_r-1}(S) - r + 1 && \text{by Theorem 2.3}\\
&\leq \sum_{i=1}^{r-1}[\alpha_S(s-1)(2t_i-3)+2]\\
&\quad + [\alpha_S(s-1)(t_r-1)+2] - r + 1 && \text{by Theorem 3.1}\\
&\leq \alpha_S(s-1)\left[\sum_{i=1}^{r}(2t_i-3)\right] + 2r - r + 1\\
&\leq \alpha_S(s-1)[t(r+1)-3r] + r + 1 && \text{by Lemma 3.2}\\
&\leq \alpha_S(s-1)t(r+1) - 3\alpha_S(s-1)r + r + 1\\
&\leq \alpha_S(s-1)t(r+1) && \text{since } s > 1 \text{ and } \alpha_S \geq 1
\end{aligned}$$

as desired. Finally, recall that $\alpha_S \leq 2$ by Proposition 3.2. ■

We conclude this section by summarizing for trees the bounds obtained so far.

Theorem 3.3. *Let S be a tree with s edges, and let T be a tree with t edges. Then*

$$1 + (s-1)(t-2)/2 \leq f(S,T) \leq (s-1)(t^2+3t).$$

Furthermore, if S is a tree for which the Erdős–Sós conjecture holds then

$$f(S,T) \leq (1/2)(s-1)(t^2+3t).$$

If t is sufficiently large and congruent to 2 *modulos, then*

$$(s-1)(t-1)+2 \leq f(S,T).$$

Proof. The lower bounds come by taking $G = S$ and $H = T$ in Theorems 2.1 and 2.2 and noting that $p = s+1$ and $k = t$ in this case. The upper bound comes from Theorem 3.2 by recalling that $r \leq (t+1)/2$, $\alpha_S \leq 2$, and $\alpha_S \leq 1$ if the Erdős–Sós conjectures holds for S. ■

4. *S* IS A STAR

In this section, we focus on the case, when S is a star. We first give a general lower bound on $f(K_{1,s}, T)$, when T is any graph.

Theorem 4.1. *Let* T *be a graph with* t *edges. Then* $f(K_{1,s}, T) \geq (s-1)(t-1)+1$.

Proof. Let $n=(s-1)(t-1)+1$. We define an edge-coloring of $K=K_n$ that contains neither a monochromatic copy of $K_{1,s}$ nor a rainbow copy of T. Since $\chi'(K) \leq n$. There exists a decomposition of $E(K_n)$ into n matchings $F_1, F_2, \ldots, F_n$. By grouping these n matchings into $(t-1)$ groups of size $s-1$, we obtain an edge-decomposition of K into $t-1$ subgraphs $H_1, \ldots, H_{t-1}$ each of maximum degree at most $s-1$.

Now, define the coloring c of $E(K_n)$ by assigning color i to the edges of H_i. Since each H_i has maximum degree at most $s-1$, c has no monochromatic star with s edges. Since altogether only $t-1$ colors are used, there is no rainbow T. ■

When T is a tree, we have the following general upper bound on $f(K_{1,s}, T)$.

Theorem 4.2. *If* T *is a tree with* t *edges, then* $f(K_{1,s}, T) \leq s(t-1)+1$.

Proof. We use induction on t. When $t=1$, the claim holds trivially. For the induction step, let $n \geq s(t-1)+1$, and let c be any coloring of $E(K_n)$. We prove that c contains a monochromatic $K_{1,s}$ or a rainbow copy of T. Let v be leaf of T, and let u be the unique neighbor of v in T. Let $T'=T-v$; T' is a subtree of T with $t-1$ edges. We may assume that c has no monochromatic copy of $K_{1,s}$. Since $n \geq s(t-2)+1$, by induction hypothesis, c has a rainbow copy of T'. Suppose, without loss of generality that the colors used on T' are $1, \ldots, t-1$, and that color 1 is used on some edge in T' incident to u. If $c(ux) \notin [t-1]$ for some $x \in V(K_n)-V(T')$, then we can extend T' to a rainbow copy of T using edge ux. For each $j \in [t-1]$, let $A_j = \{x \in V(K_n)-V(T') : c(ux)=j\}$. Then $V(K_n)-V(T') = \bigcup_{j\in[t-1]} A_j$. Since $n \geq s(t-1)+1$, we have $|\bigcup_{i\in[t-1]} A_j| = |V(K_n)-V(T')| \geq s(t-1)+1-t = (s-1)(t-1)$. By the pigeonhole principle, we have either $|A_1| \geq s-1$, or $|A_j| \geq s$ for some $j \in \{2, \ldots, t-1\}$. In either case, we get a monochromatic copy of $K_{1,s}$. ■

Theorems 4.1 and 4.2 together imply.

Corollary 4.1. *Let* T *be a tree with* t *edges, then* $(s-1)(t-1)+1 \leq f(K_{1,s}, T) \leq s(t-1)+1$.

When T is a star with t edges, a simple application of the pigeonhole principle yields $f(K_{1,s}, K_{1,t}) \leq (s-1)(t-1)+2$. By Theorem 4.1, hence we have $f(K_{1,s}, K_{1,t}) = (s-1)(t-1)+2$, except that when s is even and t is odd the value may be smaller than this by 1. In [15], $f(K_{1,s}, K_{1,t})$ was determined exactly.

Proposition 4.1 ([15]). $f(K_{1,s}, K_{1,t}) = (s-1)(t-1)+2$.

5. *T* IS A STAR $K_{1,t}$

In this section, we use some results and techniques from design theory to obtain sharp bounds or exact values for $f(S, K_{1,t}) = R_{loc}^{t-1}(G)$ in various cases. For convenience, we call a graph that contains no subgraph isomorphic to a given graph H an *H-free graph.* When $T = K_{1,t}$, we can interpret $f(S, K_{1,t})$ as the maximum value of n such that there exists a decomposition of $E(K_{n-1})$ into S-free subgraphs such that each vertex is contained in at most $t-1$ of the subgraphs.

The following proposition is a re-statement of Corollary 3.1.

Proposition 5.1. *Let S be a tree with s edges. Then $f(S, K_{1,t}) \leq 2(s-1)(t-1)+2$. Furthermore, if the Erdős–Sós conjecture holds for S, then $f(S, K_{1,t}) \leq (s-1)(t-1)+2$.*

The next theorem is a variation of Theorem 2.2 and establishes the sharpness of the upper bound in Proposition 5.1 for infinitely many pairs s, t when S satisfies the Erdős–Sós conjecture.

Theorem 5.1. *Let S be any tree on s edges. If t is sufficiently large and $t \equiv 1, 2 \pmod{s}$, then $f(S, K_{1,t}) \geq (s-1)(t-1)+2$.*

Proof. Let $n = (s-1)(t-1)+1$. Since $t \equiv 1, 2 \pmod{s}$, we have $n(n-1) \equiv 0 \pmod{s(s-1)}$. For sufficiently large t, n is sufficiently large. By [5], there exists a decomposition of $E(K_n)$ into K_s's. Clearly, each vertex in K_n is contained in $(n-1)/(s-1) = t-1$ copies of K_s in such a decomposition. Now, color $E(K_n)$ by assigning color i to the ith copy of K_s in such a decomposition. Such a coloring contains no rainbow copy of $K_{1,t}$ by our discussion above and no monochromatic copy of S, since each color class induces a K_s, while S is a tree with $s+1$ vertices. This shows that $f(S, K_{1,t}) \geq (s-1)(t-1)+2$. ■

Corollary 3.1 and Theorem 5.1 yield the following statement, which was also obtained independently in [6].

Corollary 5.1. *If S is a tree for which the Erdős–Sós conjecture holds, and $t \equiv 1, 2 \pmod{s}$ and is sufficiently large, then $f(S, K_{1,t}) = (s-1)(t-1)+2$.*

Next, we consider $f(S, K_{1,t})$ specifically, when S is the path P_{s+1} with $s+1$ vertices. Recall that the Erdős–Sós Conjecture holds for paths. In fact, that conjecture was motivated by the following result due to Erdős and Gallai [10] on Turán numbers of paths.

Theorem 5.2 (Erdős–Gallai [10]). *Given n, s, we have $ex(n, P_{s+1}) \leq n(s-1)/2$. Furthermore, equality holds only when $n = ms$, for some positive integer m, in which case, mK_s is the unique extremal graph.*

In the notation and terminology of Section 3, we thus have $\alpha_S = 1$ for any path $S = P_{s+1}$ and K_s the only *connected* strongly P_{s+1}-extremal graph. Thus, Theorem 3.1 implies the following upper bound for paths.

Corollary 5.2. *For any path P_{s+1} on s edges and star $K_{1,t}$ on t edges,*

$$f(P_{s+1}, K_{1,t}) \leq (s-1)(t-1) + 2$$

with equality if and only if there exists a decomposition of $K_{(s-1)(t-1)+1}$ into K_s's with exactly $t-1$ of the K_s's at each vertex.

We now determine $f(P_n, K_{1,t})$ exactly when $n = 4, 5$ with the aid of the following result from combinatorial design theory.

Lemma 5.1 [2]. *There exists an edge decomposition of K_n into K_3 iff $n \equiv 1, 3 \pmod 6$, and there exists an edge decomposition of K_n into K_4 iff $n \equiv 1, 4 \pmod{12}$.*

The following theorem on $f(P_4, K_{1,t})$ was obtained independently in [15]; we include our proof for completeness.

Theorem 5.3. *$f(P_4, K_{1,t}) = 2t$ if $t \equiv 1, 2 \pmod 3$ and $f(P_4, K_{1,t}) = 2t - 1$ if $t \equiv 0 \pmod 3$.*

Proof. By Corollary 5.2, we have $f(P_4, K_{1,t}) \leq 2(t-1) + 2$, with equality iff $E(K_{2t-1})$ can be decomposed into triangles. Such a decomposition exists iff $t \equiv 1, 2 \pmod 3$ by Lemma 5.1. Hence, we have $f(P_4, K_{1,t}) = 2t$ when $t \equiv 1, 2 \pmod 3$ and $f(P_4, K_{1,t}) \leq 2t - 1$ when $t \equiv 0 \pmod 3$.

To complete the proof, we need to show that $f(P_4, K_{1,t}) \geq 2t - 1$ when $t \equiv 0 \pmod 3$. It suffices to show that K_{2t-2} can be edge-decomposed into subgraphs, with each subgraph being a triangle or a star (note that triangles and stars are the only connected P_4-free graphs), such that each vertex is contained in at most $t-1$ subgraphs. Let v be a vertex in K_{2t-2}, then $K_{2t-2} - v$ is a complete graph on $2t-3$ vertices. Since $t \equiv 0 \pmod 3$, we have $2t - 3 \equiv 3 \pmod 6$ and by Lemma 5.1, the edges of $K_{2t-2} - v$ can be decomposed into triangles. We complete our decomposition by taking the star with $2t-3$ edges centered at v. It is easy to verify that this decomposition satisfies the requirements. ■

Theorem 5.4. *$f(P_5, K_{1,t}) = 3t - 1$ when $t \equiv 1, 2 \pmod 4$ and $f(P_5, K_{1,t}) = 3t - 2$ when $t \equiv 0, 3 \pmod 4$ except possibly when $t = 4$.*

Proof. By Corollary 5.2, we have $f(P_5, K_{1,t}) \leq 3t - 1$ where equality holds iff there exists an edge decomposition of K_{3t-2} into K_4's such that each vertex is in at most $t-1$ of the K_4's. Such a decomposition is known to exist iff $t \equiv$

$1,2 \pmod 4$ by Lemma 5.1. Hence, we have $f(P_5, K_{1,t}) = 3t-1$ when $t \equiv 1,2 \pmod 4$, and $f(P_5, K_{1,t}) \leq 3t-2$ when $t \equiv 0,3 \pmod 4$. Next, we show that $f(P_5, K_{1,t}) \geq 3t-2$ when $t \equiv 0,3 \pmod 4$ except possibly when $t=4$. We do so by giving a coloring to $E(K_{3t-3})$ for the above values of t that has no monochromatic copy of P_5 or a rainbow copy of $K_{1,t}$.

When we cover the edges of K_{3t-3} using K_4's, we need at least $\lceil \frac{3t-4}{3} \rceil$ K_4's to cover edges incident to each vertex, hence altogether $\lceil \frac{3t-3}{4} \lceil \frac{3t-4}{3} \rceil \rceil$ K_4's are needed, with equality only if each vertex is in exactly $\lceil \frac{3t-4}{3} \rceil = t-1$ of the K_4's. When $t \equiv 0,3 \pmod 4$ except $t=4$, it is known ([23]) that K_{3t-3} can be covered by a collection of $\lceil \frac{3t-3}{4} \lceil \frac{3t-4}{3} \rceil \rceil$ K_4's. By our discussion above, each vertex is in exactly $t-1$ K_4's in such a covering. We can use this covering to obtain a decomposition of $E(K_{3t-3})$ into subgraphs of order 4, such that each vertex belongs to $t-1$ of the subgraphs. The decomposition naturally corresponds to a coloring of $E(K_{3t-3})$, which has no monochromatic copy of P_5 or rainbow copy of $K_{1,t}$. ■

For all positive $t \geq 3$, let B_t denote the tree with t edges obtained by adding a pendant edge to a leaf of $K_{1,t-1}$. The techniques used in this section can be used to obtain some exact values of $f(S,T)$, when $T = B_t$. Due to the length constraint of the paper, we only list the results here without proofs. Interested readers are encouraged to contact the authors for details.

Proposition 5.2. *For any graph G, we have* $f(G, B_t) \leq f(G, K_{1,t})$. *In particular, if S is a tree on s edges, then* $f(S, B_t) \leq 2(s-1)(t-1)+2$. *Furthermore, if S satisfies the Erdős–Sós conjecture then* $f(S, B_t) \leq (s-1)(t-1)+2$.

Corollary 5.3. *If S is a tree on s edges for which the Erdős–Sós conjecture holds, and* $t \equiv 2 \pmod s$ *and is sufficiently large, then* $f(S, B_t) = (s-1)(t-1)+2$.

Proposition 5.3. *We have* $f(P_4, B_t) = 2t$ *if* $t \equiv 2 \pmod 3$, *and* $f(P_4, B_t) = 2t-1$ *otherwise except possibly when* $t = 4, 7$.

6. CONCLUDING REMARKS

Based on the values of f obtained, we think it is quite possible that there exists an absolute constant α such that for any pair of trees S, T with s and t edges, respectively, one has $f(S,T) \leq \alpha(s-1)(t-1)$, which, if true, would be sharp up to a multiplicative constant. Also, we ask the following question.

Question 6.1. Among all pairs of trees S, T with s edges and t edges, respectively, is $f(S,T)$ maximized by $f(P_{s+1}, P_{t+1})$?

In general, since $f(G,H)$ exists iff G is a star or H is acyclic, this points out two natural directions of further study: (1) study $f(G,H)$ when G is a fixed star and H is any graph and (2) study $f(G,H)$ when G is any graph and H is a fixed forest. For a study along direction (1), see [4]. A study along direction (2) may be more closely related to local Ramsey numbers.

NOTE

After the manuscript was written, we learned that Chen, Schelp, and Wei [8] had independently considered essentially the same problem for trees. Eroh and Oellerman [11] have considered a bipartite analog of the problem.

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