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EDGE-BANDWIDTH OF GRAPHS*

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Abstract. The *edge-bandwidth* of a graph is the minimum, over all labelings of the edges with distinct integers, of the maximum difference between labels of two incident edges. We prove that edge-bandwidth is at least as large as bandwidth for every graph, with equality for certain caterpillars. We obtain sharp or nearly sharp bounds on the change in edge-bandwidth under addition, subdivision, or contraction of edges. We compute edge-bandwidth for K_n, $K_{n,n}$, caterpillars, and some theta graphs.

Key words. bandwidth, edge-bandwidth, clique, biclique, caterpillar

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1. Introduction.

A classical optimization problem is to label the vertices of a graph with distinct integers so that the maximum difference between labels on adjacent vertices is minimized. For a graph G, the optimal bound on the differences is the *bandwidth* $B(G)$. The name arises from computations with sparse symmetric matrices, where operations run faster when the matrix is permuted so that all entries lie near the diagonal. The bandwidth of a matrix M is the bandwidth of the corresponding graph whose adjacency matrix has a 1 in those positions where M is nonzero. Early results on bandwidth are surveyed in [2] and [3].

In this paper, we introduce an analogous parameter for edge-labelings. An *edge-numbering* (or *edge-labeling*) of a graph G is a function f that assigns distinct integers to the edges of G. We let $B'(f)$ denote the maximum of the difference between labels assigned to adjacent (incident) edges. The *edge-bandwidth* $B'(G)$ is the minimum of $B'(f)$ over all edge-labelings. The term "edge-numbering" is used because we may assume that f is a bijection from $E(G)$ to the first $|E(G)|$ natural numbers.

We use the notation $B'(G)$ for the edge-bandwidth of G because it is immediate that the edge-bandwidth of a graph equals the bandwidth of its line graph. Thus well-known elementary bounds on bandwidth can be applied to line graphs to obtain bounds on edge-bandwidth. We mention several such bounds. We compute edge-bandwidth on a special class where all these bounds are arbitrarily bad.

The relationship between edge-bandwidth and bandwidth is particularly interesting. Always $B(G) \leq B'(G)$, with equality for caterpillars of diameter more than k in which every vertex has degree 1 or $k+1$. Among forests, $B'(G) \leq 2B(G)$, which is almost sharp for stars. More generally, if G is a union of t forests, then $B'(G) \leq 2tB(G) + t - 1$.

Chvátalová and Opatrný [5] studied the effect on bandwidth of edge addition, contraction, and subdivision (see [22] for further results on edge addition). We study

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these for edge-bandwidth. Adding or contracting an edge at most doubles the edge-bandwidth. Subdividing an edge decreases the edge-bandwidth by at most a factor of 1/3. All these bounds are sharp within additive constants. Surprisingly, subdivision can also increase edge-bandwidth, but at most by 1, and contraction can decrease it by 1.

Because the edge-bandwidth problem is a restriction of the bandwidth problem, it may be easier computationally. Computation of bandwidth is NP-complete [17], remaining so for trees with maximum degree 4 [8] and for several classes of caterpillar-like graphs [11, 16]. Such graphs generally are not line graphs (they contain claws). It remains open whether computing edge-bandwidth (computing bandwidth of line graphs) is NP-hard.

Due to the computational difficulty, bandwidth has been studied on various special classes. Bandwidth has been determined for caterpillars and for various generalizations of caterpillars [1, 11, 14, 21], for complete k-ary trees [19], for rectangular and triangular grids [4, 10] (higher dimensions [9, 15]), for unions of pairwise internally disjoint paths with common endpoints (called "theta graphs" [6, 13, 18]), etc. Polynomial-time algorithms exist for computing bandwidth for graphs in these classes and for interval graphs [12, 20]. We begin analogous investigations for edge-bandwidth by computing the edge-bandwidth for cliques, for equipartite complete bipartite graphs, and for some theta graphs.

2. Relation to other parameters. We begin by listing elementary lower bounds on edge-bandwidth that follow from standard arguments about bandwidth when applied to line graphs.

PROPOSITION 2.1. *Edge-bandwidth satisfies the following:*
(a) $B'(H) \leq B'(G)$ *when* H *is a subgraph of* G.
(b) $B'(G) = \max\{B'(G_i)\}$, *where* $\{G_i\}$ *are the components of* G.
(c) $B'(G) \geq \Delta(G) - 1$.

Proof. (a) A labeling of G contains a labeling of H. (b) Concatenating labelings of the components achieves the lower bound established by (a). (c) The edges incident to a single vertex induce a clique in the line graph. The lowest and highest among these labels are at least $\Delta(G) - 1$ apart. ∎

PROPOSITION 2.2. $B'(G) \geq \max_{H \subseteq G} \left\lceil \frac{e(H)-1}{\text{diam }(L(H))} \right\rceil$.

Proof. This is the statement of Chung's "density bound" [3] for line graphs. Every labeling of a graph contains a labeling of every subgraph. In a subgraph H, the lowest and highest labels are at least $e(H) - 1$ apart, and the edges receiving these labels are connected by a path of length at most diam $(L(H))$, so by the pigeonhole principle some consecutive pair of edges along the path have labels differing by at least $(e(H) - 1)/\text{diam }(L(H))$. ∎

Subgraphs of diameter 2 include stars, and a star in a line graph is generated from an edge of G with its incident edges at both endpoints. The size of such a subgraph is at most $d(u) + d(v) - 1$, yielding the bound $B'(G) \geq [d(u) + d(v)]/2 - 1$ for $uv \in E(G)$. This is at most $\Delta(G) - 1$, the lower bound from Proposition 2.1. Nevertheless, because of the way in which stars in line graphs arise, they can yield a better lower bound for regular or nearly regular graphs. We develop this next.

PROPOSITION 2.3. *For* $F \subseteq E(G)$, *let* $\partial(F)$ *denote the set of edges not in* F *that are incident to at least one edge in* F. *The edge-bandwidth satisfies* $B'(G) \geq \max_k \min_{|F|=k} |\partial(F)|$.

Proof. This is the statement of Harper's "boundary bound" [9] for line graphs.

Some set F of k edges must be the set given the k smallest labels. If m edges outside this set have incidents with this set, then the largest label on the edges of ∂F is at least $k+m$, and the difference between the labels on this and its incident edge in F is at least m. □

COROLLARY 2.4. $B'(G) \geq \min_{uv \in E(G)} d(u)+d(v)-2$.

Proof. We apply Proposition 2.3 with $k=1$. Each edge uv is incident to $d(u)+d(v)-2$ other edges. Some edge must have the least label, and this establishes the lower bound. □

Although these bounds are often useful, they can be arbitrarily bad. The *theta graph* $\Theta(l_1,\ldots,l_m)$ is the graph that is the union of m pairwise internally disjoint paths with common endpoints and lengths $l_1,\ldots,l_m$. The name "theta graph" comes from the case $m=3$. The bandwidth is known for all theta graphs, but settling this was a difficult process finished in [18]. When the path lengths are equal, the edge-bandwidth and bandwidth both equal m, using the density lower bound and a simple construction. The edge-bandwidth can be much higher when the lengths are unequal. Our example showing this will later demonstrate sharpness of some bounds.

Our original proof of the lower bound was lengthy. The simple argument presented here originated with Dennis Eichhorn and Kevin O'Bryant. It will be generalized in [7] to compute edge-bandwidth for a large class of theta graphs.

Example A. Consider $G=\Theta(l_1,\ldots,l_m)$ with $l_m=1$ and $l_1=\cdots=l_{m-1}=3$. Let a_i,b_i,c_i denote the edges of the ith path of length 3, and let e be the edge incident to all a_i's at one end and to all c_i's at the other end. Since $\Delta(G)=m$, Proposition 2.1(c) yields $B'(G)\geq m-1$. Proposition 2.2 also yields $B'(G)\geq m-1$. For $1\leq k\leq 2m-2$, the first k edges in the list $a_1,\ldots,a_{m-1},b_1,\ldots,b_{m-1}$ are together incident to exactly m other edges, and larger sets are incident to at most $m-1$ other edges. Thus the best lower bound from Proposition 2.3 is at most m.

Nevertheless, $B'(G)=\lceil(3m-3)/2\rceil$. For the upper bound, we assign the $3m-2$ labels in order to a's, b's, and c's, inserting e before $b_{\lceil m/2\rceil}$. The difference between labels of incidence edges is always at most m except for incidences involving e, which are at most $\lceil(3m-3)/2\rceil$ since e has the middle label.

$$a_1,\ldots,a_{m-1},b_1,\ldots,b_{\lceil m/2\rceil-1},e,b_{\lceil m/2\rceil},\ldots,b_{m-1},c_1,\ldots,c_{m-1}$$

To prove the lower bound, consider a numbering f of $E(G)$ by distinct integers and let $k=B'(f)$. Let $\alpha=\max\{f(e),\max_i\{f(a_i)\}\}$ and $\alpha'=\min\{f(e),\min_i\{f(c_i)\}\}$. Comparing the edges with labels $\alpha, f(e), \alpha'$ yields $\alpha-k\leq f(e)\leq\alpha'+k$. Let I be the interval $[\alpha-k,\alpha'+k]$. By construction, I contains the labels of all a's, all c's, and e. If $f(a_i)<\alpha'$ and $f(c_i)>\alpha$, then also $f(b_i)\in I$. By the choice of α,α', avoiding this requires $\alpha'<f(a_i)\leq\alpha$ or $\alpha'\leq f(c_i)<\alpha$. Since each label is assigned only once and the label $f(e)$ cannot play this role, only $\alpha-\alpha'$ of the b's can have labels outside I. Counting the labels we have forced into I yields $|I|\geq(2m-1)+(m-1-\alpha+\alpha')$. On the other hand, $|I|=2k+\alpha'-\alpha+1$. Thus $k\geq(3m-3)/2$, as desired. □

3. Edge-bandwidth vs. bandwidth. In this section we prove various best-possible inequalities involving bandwidth and edge-bandwidth. The proof that $B(G)\leq B'(G)$ requires several steps. All steps are constructive. When f or g is a labeling of the edges or vertices of G, we say that $f(e)$ of $g(v)$ is the *f-label* or *g-label* of the edge e or vertex v. An *f-label* on an edge incident to u is an *incident f-label* of u.

LEMMA 3.1. *If a finite graph G has minimum degree at least two, then $B(G)\leq B'(G)$.*

Proof. From an optimal edge-numbering f (such that $B'(f) = B'(G) = m$), we define a labeling g of the vertices. The labels used by g need not be consecutive, but we show that $|g(u) - g(v)| \leq m$ when u and v are adjacent.

We produce g in phases. At the beginning of each phase, we choose an arbitrary unlabeled vertex u and call it the *active vertex*. At each step in a phase, we select the unused edge e of smallest f-label among those incident to the active vertex. We let $f(e)$ be the g-label of the active vertex, mark e *used*, and designate the other endpoint of e as the active vertex. If the new active vertex already has a label, we end the phase. Otherwise, we continue the phase.

When we examine a new active vertex, it has an edge with least incident label, because every vertex has degree at least 2 and we have not previously reached this vertex. Each phase eventually ends, because the vertex set is finite and we cannot continue reaching new vertices. The procedure assigns a label $g(u)$ for each $u \in V(G)$, since we continue to a new phase as long as an unlabeled vertex remains.

It remains to verify that $|g(u) - g(v)| \leq m$ when $uv \in E(G)$. Suppose that $g(u) = a = f(e)$ and $g(v) = b = f(e')$. Since each vertex is assigned the f-label of an incident edge, we have e, e' incident to u, v, respectively. If the edge uv is one of e, e', then e and e' are incident, which implies that $|g(u) - g(v)| = |f(e) - f(e')| \leq m$.

Otherwise, we have $f(uv) = c$ for some other value c. We may assume that $a < b$ by symmetry. If $a < c$ and $b < c$, then $|g(u)-g(v)| = b-a < c-a = f(uv)-f(e) \leq m$. Thus we may assume that $b > c$. In particular, $g(v)$ is not the least f-label incident to v.

The algorithm assigns v a label when v first becomes active, using the least f-label among *unused* incident edges. When v first becomes active, only the edge of arrival is a used incident edge. Thus $g(v)$ is the least incident f-label except when v is first reached via the least-labeled incident edge. In this case, $g(v)$ is the second smallest incident f-label. Thus c is the least f-label incident to v and v becomes active by arrival from u. This requires $g(u) = c$, which contradicts $g(u) = a$ and eliminates the bad case. ∎

LEMMA 3.2. *If G is a tree, then $B(G) \leq B'(G)$.*

Proof. Again we use an optimal edge-numbering f to define a vertex-labeling g whose adjacent vertices differ by at most $B'(f)$. We may assume that the least f-label is 1, occurring on the edge $e = uv$. Assign (temporarily) $g(u) = g(v) = f(e)$. View the edge e as the root of G. For each vertex $x \notin \{u, v\}$, let $g(x)$ be the f-label of the edge incident to x along the path from x to the root.

If $xy \in E(G)$ and $xy \neq uv$, then we may assume that y is on the path from x to the root. We have assigned $g(x) = f(xy)$, and $g(y)$ is the f-label of an edge incident to y, so $|g(x) - g(y)| \leq B'(f)$.

Our labeling g fails to be the desired labeling only because we used 1 on both u and v. Observe that the largest f-label incident to uv occurs on an edge incident to u or on an edge incident to v but not both; we may assume the latter. Now we change $g(u)$ to 0. Because the differences between $f(uv)$ and f-labels on edges incident to u were *less* than $B'(f)$, this produces the desired labeling g. ∎

THEOREM 3.3. *For every graph G, $B(G) \leq B'(G)$.*

Proof. By Proposition 2.1(b), it suffices to consider connected graphs. Let f be an optimal edge-numbering of G; we produce a vertex labeling g. Lemma 3.2 applies when G is a tree. Otherwise, G contains a cycle, and iteratively deleting vertices of degree 1 produces a subgraph G' in which every vertex has degree at least 2. The algorithm of Lemma 3.1, applied to the restriction of f to G', produces a vertex labeling g of G' in which (1) adjacent vertices have labels differing by at most $B'(f)$, and (2) the label on each vertex is the f-label of some edge incident to it in G'.

To obtain a vertex labeling of G, reverse the deletion procedure. This iteratively adds a vertex x adjacent to a vertex y that already has a g-label. Assign to x the f-label of the edge xy in the full edge-numbering f of G. Now $g(x)$ and $g(y)$ are the f-labels of two edges incident to y in G, and thus $|g(x) - g(y)| \leq B'(f)$. The claims (1) and (2) are preserved, and we continue this process until we replace all vertices that were deleted from G. □

A *caterpillar* is a tree in which the subtree obtained by deleting all leaves is a path. One of the characterizations of caterpillars is the existence of a linear ordering of the edges such that each prefix and each suffix forms a subtree. We show that such an ordering is optimal for edge-bandwidth and use this to show that Theorem 3.3 is nearly sharp.

PROPOSITION 3.4. *If G is a caterpillar, then $B'(G) = \Delta(G) - 1$. Let G be the caterpillar of diameter d in which every vertex has degree $k+1$ or 1. If $d \geq k$, then $B(G) = B'(G) = k$.*

Proof. Let G be a caterpillar. Let $v_1, \dots, v_{d-1}$ be the nonleaf vertices of the dominating path. The diameter of G is d. Number the edges by assigning labels in the following order: first the pendant edges incident to v_1, then v_1v_2, then the pendant edges incident to v_2, then v_2v_3, etc. Since edges are incident only at $v_1, \dots, v_{d-1}$, this ordering places all pairs of incident edges within $\Delta(G) - 1$ positions of each other. Since $B'(G) \geq \Delta(G) - 1$ for all G, equality holds.

For a caterpillar G with order n and diameter d, Chung's density bound yields $B(G) \geq (n-1)/d$. Let G be the caterpillar of diameter d in which every vertex has degree $k+1$ or 1. We have $d-1$ vertices of degree $k+1$, so $n = (d-1)k + 2$ and $B(G) > k - k/d$. When $d \geq k$, we have $B(G) \geq k$.

On the other hand, we have observed that $B'(G) \leq \Delta(G) - 1 = k$ for caterpillars. By Theorem 3.3, equality holds throughout for these special caterpillars. □

Theorem 3.3 places a lower bound on $B'(G)$ in terms of $B(G)$. We next establish an upper bound. The *arboricity* is the minimum number of forests needed to partition the edges of G.

THEOREM 3.5. *If G has arboricity t, then $B'(G) \leq 2tB(G) + t - 1$. When $t = 1$, the inequality is almost sharp; there are caterpillars with $B'(G) = 2B(G) - 1$.*

Proof. Given an optimal number g of $V(G)$, we construct a labeling f of $E(G)$. Let $G_1, \dots, G_t$ be a decomposition of G into the minimum number of forests. In each component of each G_i, select a root. Each edge of G_i is the first edge on the path from one of its endpoints to the root of its component in G_i; for $e \in E(G_i)$, let $v(e)$ denote this endpoint. Define $f(e) = tg(v(e)) + i$.

Each vertex of each forest heads toward the root of its component in that forest along exactly one edge, so the f-labels of the edges are distinct. Each f-label arises from the g-label of one of its endpoints. Thus the f-labels of two incident edges arise from the g-labels of vertices separated by distance at most 2 in G. Also, the indices of the forests containing these edges differ by at most $t-1$. Thus when e, e' are incident we have $|f(e) - f(e')| \leq t2B(g) + t - 1$.

The bandwidth of a caterpillar is the maximum density (number of edges divided by diameter) over subtrees [14]. This equals $\lceil \Delta(G)/2 \rceil$ whenever the vertex degrees all lie in $\{\Delta(G), 2, 1\}$ and the vertices of degree $\Delta(G)$ are pairwise. (Without [14], this still holds explicitly for stars.) □

4. Effect of edge operations. In this section, we obtain bounds on the effect of local edge operations on the edge-bandwidth. The variations can be linear in the value of the edge-bandwidth, and our bounds are optimal except for additive constants. We study addition, subdivision, and contraction of edges.

THEOREM 4.1. *If H is obtained from G by adding an edge, then $B'(G) \le B'(H) \le 2B'(G)$. Furthermore, for odd k there are examples of $H = G+e$ such that $B'(G) = k$ and $B'(H) \ge 2k-1$.*

Proof. The first inequality holds because G is a subgraph of H. For the second, let g be an optimal edge-numbering of G; we produce an edge-numbering f of H such that $B'(f) \le 2B'(g)$.

If e is not incident to an edge of G, form f from g by giving e a new label higher than the others. If only one endpoint of e is incident to an edge e' of G, form f by leaving the g-labels less than $g(e')$ unchanged, augmenting the remaining labels by 1, and letting $f(e) = g(e')+1$. We have $B(f) \le B(g)+1$.

Thus we may assume that the new edge e joins two vertices of G. Our construction for this case modifies an argument in [22]. Let e_i be the edge such that $g(e_i) = i$, for $1 \le i \le B(g)$. Let p, q be the smallest and largest indices of edges of G incident to e, respectively, and let $r = \lfloor (p+q)/2 \rfloor$.

The idea in defining f from g is to "fold" the ordering at r, renumbering out from there so that e_p and e_q receive consecutive labels, and inserting e just before this. The renumbering of the old edges is as follows:

$$f(e_j) = \begin{cases} 2(j-r) & \text{if } r < j < q, \\ 2(j-r)+1 & \text{if } q \le j, \\ 2(r-j)+1 & \text{if } p < j \le r, \\ 2(r-j)+2 & \text{if } j \le p. \end{cases}$$

Finally, let $f(e) = \min\{f(e_p), f(e_q)\} - 1 = q-p$. After the edges with g-labels higher than q or lower than p are exhausted, the new numbering leaves gaps. For edges $e_i, e_j \in E(G)$, we have $|f(e_i) - f(e_j)| \le 2|i-j|+1$, where the possible added 1 stems from the insertion of e. When r is between i and j, the actual stretch is smaller.

It remains to consider incidences involving e. Suppose that $e' = e_j$ is incident to e. Note that $1 \le f(e') \le q-p+2 = f(e)+2$; we may assume that $1 \le f(e') < f(e)$. If e_p and e_q are incident to the same endpoint of e, then $1 \le f(e) - f(e') \le q-p+1 \le B(g)+1$. If e_p and e_q are incident to opposite endpoints of e, then e' is incident to e_p or e_q. In these two cases, we have $p \le j \le p + B(g)$ or $q - B(g) \le j \le q$. Since j differs from p or q, respectively, by at most $B(g)$, we obtain $1 \le f(e) - f(e') \le 2B(g)$.

The bound is nearly sharp when k is odd. Let G be the caterpillar of diameter $k+1$ with vertices of degrees $k+1$ and 1 (see Proposition 3.4). We have $e(G) = k^2+1$ and $B'(G) = B(G) = k$. The graph H formed by adding the edge $v_1 v_k$ is a cycle of length k plus pendant edges; each vertex of the cycle has degree $k+1$ except for two adjacent vertices of degree $k+2$. The diameter of $L(H)$ is $\lfloor k/2 \rfloor + 1 = (k+1)/2$, and H has k^2+2 edges. By Proposition 2.2, we obtain $B'(H) \ge \left\lceil \frac{k^2+1}{(k+1)/2} \right\rceil = \left\lceil 2k-2+\frac{4}{k+1} \right\rceil = 2k-1$. □

Subdividing an edge uv means replacing uv by a path u, w, v passing through a new vertex w. If H is obtained from G by subdividing one edge of G, then H is an *elementary subdivision* of G. Edge subdivision can reduce the edge-bandwidth considerably, but it increases the edge-bandwidth by at most one.

THEOREM 4.2. *If H is an elementary subdivision of G, then $\lceil (2B'(G)+\delta)/3 \rceil \le B'(H) \le B'(G)+1$, where δ is 1 if $B'(H)$ is odd and 0 if $B'(H)$ is even, and these* *bounds are sharp.*

Proof. Suppose that H is obtained from G by subdividing edge e. From an optimal edge-numbering g of G, we obtain an edge-numbering of H by augmenting the labels greater than $g(e)$ and letting the labels of the two new edges be $g(e)$ and $g(e)+1$. This stretches the difference between incident labels by at most 1.

For sharpness of the bound, compare $G = \Theta(1, 2, \ldots, 2)$ and $G' = \Theta(1, 3, \ldots, 3)$, where each has m paths with common endpoints. In Example A, we proved that $B'(G') = \lceil 3(m-1)/2 \rceil$. In G, let the ith path have edges a_i, b_i for $i < m$, with e the extra edge. The ordering $a_1, \ldots, a_{m-1}, e, b_1, \ldots, b_{m-1}$ yields $B'(G) \leq m$. The graph G' is obtained from G by a sequence of $m-1$ elementary subdivisions, roughly half of which must increase the edge-bandwidth. The desired graph H is the first where the bandwidth is $m+1$.

To prove the lower bound on $B'(H)$, we consider an optimal edge-numbering f of H and obtain an edge-numbering of G. For the edges e', e'' introduced to form H after deleting e, let $p = f(e')$ and $q = f(e'')$. We may assume that $p < q$. Let $r = \lfloor (p+q)/2 \rfloor$. Define g by leaving the f-labels below p and in $[r+1, q-1]$ unchanged, decreasing those in $[p+1, r]$ and above q by 1, and setting $g(e) = r$. The differences between labels on edges belonging to both G and H change by at most 1 and increase only when the difference is less than $B'(f)$. For incidents involving e, the incident edge ϵ was incident in H to e' or e''. The difference $|g(e) - g(\epsilon)|$ exceeds $B'(f)$ only if $g(\epsilon) < p$ or $g(\epsilon) > q$. In the first case, the difference increases by $r - p = \lfloor (q-p)/2 \rfloor$. In the second, it increases by $q - r - 1 = \lceil (q-p)/2 \rceil - 1$. We obtain $B'(G) \leq B'(H) + \left\lfloor \frac{q-p}{2} \right\rfloor \leq \left\lfloor \frac{3B'(H)}{2} \right\rfloor$. Whether $B'(H)$ is even or odd, this establishes the bound claimed.

For sharpness of the bound, compare $G = \Theta(1, 3, \ldots, 3)$ and $H = \Theta(2, 3, \ldots, 3)$. In H let the ith path have edges a_i, b_i, c_i for $i < m$, with d, e the remaining path. The ordering $a_1, \ldots, a_{m-1}, d, b_1, \ldots, b_{m-1}, e, c_1, \ldots, c_{m-1}$ yields $B'(H) \leq m$. From Example A, $B'(G) = \lceil 3(m-1)/2 \rceil$. Whether m is odd or even, this example achieves the lower bound on $B'(H)$. ∎

Contracting an edge uv means deleting the edge and replacing its endpoints by a single combined vertex w inheriting all other edge incidences involving u and v. Contraction tends to make a graph denser and thus increase edge-bandwidth. In some applications, one restricts attention to simple graphs and thus discards loops or multiple edges that arise under contraction. Such a convention can discard many edges and thus lead to a decrease in edge-bandwidth. In particular, contracting an edge of a clique would yield a smaller clique under this model and thus smaller edge-bandwidth.

For the next result, we say that H is an *elementary contraction* of G if H is obtained from G by contracting one edge and keeping all other edges, regardless of whether loops or multiple edges arise. Edge-bandwidth is a valid parameter for multigraphs.

THEOREM 4.3. *If H is an elementary contraction of G, then $B'(G) - 1 \leq B'(H) \leq 2B'(G) - 1$, and these bounds are sharp for each value of $B'(G)$.*

Proof. Let e be the edge contracted to produce H. For the upper bound, let g be an optimal edge-numbering of G, and let f be the edge-numbering of H produced by deleting e from the numbering. In particular, leave the g-labels below $g(e)$ unchanged and decrement those above $g(e)$ by 1. Edges incident in H have distance at most 2 in $L(G)$, and their distance in $L(G)$ is 2 only if e lies between them. Thus the difference between their g-labels is at most $2B'(g)$, with equality only if the difference between their f-labels is $2B'(G) - 1$.

Equality holds when G is the double-star (the caterpillar with two vertices of degree $k+1$ and $2k$ vertices of degree 1) and e is the central edge of G, so H is the star $K_{1,2k}$. We have observed that $B'(G) = k$ and $B'(H) = 2k - 1$.

For the lower bound, let f be an optimal edge-numbering of H, and let g be the edge-numbering of G produced by inserting e into the numbering just above the edge e' with lowest f-label among those incident to the contracted vertex w in H. In particular, leave f-labels up to $f(e')$ unchanged, augment those above $f(e')$ by 1, and let $g(e) = f(e')+1$. The construction and the argument depend on the preservation of loops and multiple edges. Edges other than e that are incident in G are also incident in H, and the difference between their labels under g is at most one more than the difference under f. Edges incident to e in G are incident to e' in H and thus have f-label at most $f(e') + B'(f)$. Thus their g-label differs from that of e' by at most $B'(f)$.

The lower bound must be sharp for each value of $B'(G)$, because successive contractions eventually eliminate all edges and thus reduce the bandwidth. □

5. Edge-bandwidth of cliques and bicliques. We have computed edge-bandwidth for caterpillars and other sparse graphs. In this section we compute edge-bandwidth for classical dense families, the cliques and equipartite complete bipartite graphs. Given the difficulty of bandwidth computations, the existence of exact formulas is of as much interest as the formulas themselves.

THEOREM 5.1. *$B'(K_n) = \lfloor n^2/4 \rfloor + \lceil n/2 \rceil - 2$.*

Proof. Lower bound. Consider an optimal numbering. Among the lowest $\binom{\lceil n/2\rceil - 1}{2} + 1$ values there must be edges involving at least $\lceil n/2 \rceil$ vertices of K_n. Among the highest $\binom{\lfloor n/2 \rfloor}{2} + 1$ values there must be edges involving at least $\lfloor n/2 \rfloor + 1$ vertices of K_n. Since $\lceil n/2 \rceil + \lfloor n/2 \rfloor + 1 > n$, some vertex has incident edges with labels among the lowest $\binom{\lceil n/2\rceil - 1}{2} + 1$ and among the highest $\binom{\lfloor n/2 \rfloor}{2} + 1$. Therefore,

$$\begin{aligned} B'(K_n) &\geq \left[\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}\right] - \left[\binom{\lceil n/2 \rceil - 1}{2} + 1\right] \\ &= \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n - 1 - 1 \\ &= \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - 2. \end{aligned}$$

Upper bound. To achieve the bound above, let X, Y be the vertex partition with $X = \{1, \ldots, \lceil n/2 \rceil\}$ and $Y = \{\lceil n/2 \rceil + 1, \ldots, n\}$. We assign the lowest $\binom{\lceil n/2 \rceil}{2}$ values to the edges within X. We use reverse lexicographic order, listing first the edges with higher vertex 2, then higher vertex 3, etc. We assign the highest $\binom{\lfloor n/2 \rfloor}{2}$ values to the edges within Y by the symmetric procedure.

u	1	1	2	1	2	3	$\cdots$	$\cdots$	$n-3$	$n-3$	$n-3$	$n-2$	$n-2$	$n-1$
v	2	3	3	4	4	4	$\cdots$	$\cdots$	$n-2$	$n-1$	n	$n-1$	n	n
$f(uv)$	1	2	3	4	5	6	$\cdots$	$\cdots$	$\binom{n}{2} - 5$					$\binom{n}{2}$

Note that the lowest label on an edge incident to vertex $\lceil n/2 \rceil$ is $1 + \binom{\lceil n/2 \rceil - 1}{2}$.

The labels between these ranges are assigned to the "cross-edges" between X and Y. The cross-edges involving the vertex $\lceil n/2 \rceil \in X$ receive the highest of the central labels, and the cross-edges involving $\lceil n/2 \rceil + 1 \in Y$ (but not $\lceil n/2 \rceil$) receive the lowest of these labels. Since the highest cross-edge label is $\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}$ and the lowest label of an edge incident to $\lceil n/2 \rceil$ is $1 + \binom{\lceil n/2 \rceil - 1}{2}$, the maximum difference between labels on edges incident to $\lceil n/2 \rceil$ is precisely the lower bound on $B'(K_n)$ computed above. This observation holds symmetrically for the edges incident to $\lceil n/2 \rceil + 1$.

1 1 2 1 2 3 5 5 5 6 6 7 8 7 8 6 7 8 5 6 7 8 5 5 5 6 6 7
2 3 3 4 4 4 1 2 3 1 2 1 1 2 2 3 3 3 4 4 4 4 6 7 8 7 8 8

We now procede iteratively. On the high end of the remaining gap, we assign the values to the remaining edges incident to $\lceil n/2 \rceil - 1$. Then on the low end, we assign values to the remaining edges incident to $\lceil n/2 \rceil + 2$. We continue alternating between the top and the bottom, completing the edges incident to the more extreme labels as we approach the center of the numbering. We have illustrated the resulting order for K_8. Each time we insert the remaining edges incident to a vertex of X, the rightmost extreme moves toward the center at least as much from the previous extreme as the leftmost extreme moves toward the left. Thus the bound on the difference is maintained for the edges incident to each vertex. The observation is symmetric for edges incident to vertices of Y. ◻

For equipartite complete bipartite graphs, we have a similar construction involving low vertices, high vertices, and cross-edges.

THEOREM 5.2. $B'(K_{n,n}) = \binom{n+1}{2} - 1$.

Proof. Lower bound. We use the boundary bound of Proposition 2.3 with $k = \lfloor n^2/4 \rfloor + 1$. Every set of k edges is together incident to at least $n+1$ vertices, since a bipartite graph with n vertices has at most $k-1$ edges. Since $K_{n,n}$ has $2n$ vertices, at most $\lfloor (n-1)^2/4 \rfloor$ edges remain when these vertices are deleted. Thus when $|F| = k$, we have

$$B'(K_{n,n}) \geq |\partial(F)| \geq n^2 - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lfloor \frac{n^2}{4} \right\rfloor - 1 = \binom{n+1}{2} - 1.$$

We construct an ordering achieving this bound. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be the partite sets. Order the vertices as $L = x_1, y_1, \ldots, x_n, y_n$. We alternately finish a vertex from the beginning of L and a vertex from the end. When finishing a vertex from the beginning, we place its incident edges to vertices earlier in L at the end of the initial portion of the numbering f that has already been determined. When finishing a vertex from the end of L, we place its incident edges to vertices later in L at the beginning of the terminal portion of f that has been determined. We do not place an edge twice. When we have finished each vertex in each direction, we have placed all edges in the numbering. For example, this produces the following edge ordering for $K_{6,6}$:

X 1 2 1 2 3 3 1 2 3 4 4 4 1 2 5 1 1 6 2 2 5 6 3 3 3 5 6 4 4 4 5 6 5 5 6 6
Y 1 1 2 2 1 2 3 3 3 1 2 3 4 4 1 5 6 1 5 6 2 2 4 5 6 3 3 4 5 6 4 4 5 6 5 6

It suffices to show that for the jth vertex $v_j \in L$, there are at least $n^2 - \binom{n+1}{2} = \binom{n}{2}$ edges that come before the first edge incident to v or after the last edge incident to v. For $j = n+1$, there are exactly $\lfloor n^2/4 \rfloor$ edges before the first appearance of v_j and exactly $\lfloor (n-1)^2/4 \rfloor$ edges after its last appearance, which matches the argument in the lower bound. As j decreases, the leftmost appearance of v_j moves leftward no more quickly than the rightmost appearance; we omit the numerical details. The symmetric argument applies for $j \geq n$. □

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