COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **20** (1) (2000) 141–146

NOTE

NEW UPPER BOUNDS FOR A CANONICAL RAMSEY PROBLEM

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Received July 2, 1998

Let $f(l,k)$ be the minimum n with the property that every coloring $c\colon \binom{[n+1]}{2} \to \{1,2,\dots\}$ yields either $x_0 < \cdots < x_l$ with $c(x_0,x_1) = \cdots = c(x_{l-1},x_l)$, or $y_0 < \cdots < y_k$ with $c(y_0,y_1),\dots,c(y_{k-1},y_k)$ all distinct. We prove that if $k = o(\sqrt{l}\ )$, then $f(l,k) \sim l^{k-1}$ as $l \to \infty$. This supports the conjecture of Lefmann, Rödl, and Thomas that $f(l,k) = l^{k-1}$.

1. Canonical colorings

Ramsey Theory studies the monochromatic subgraphs that are forced to appear in every k-coloring of the edges of K_n. If we relax the requirement on the number of available colors, we can still ask what types of subgraphs are forced. Erdős and Rado [1] formulated this question more precisely. Let $\mathbb{N}$ be the set of positive integers. If $S \subseteq \mathbb{N}$ and $k>0$, then let $\binom{S}{k}$ be the family of all k-subsets of S.

Theorem 1. **(Erdős–Rado [1])** *Suppose that* $\binom{\mathbb{N}}{2}$ *is arbitrarily colored. Then there is a sequence of positive integers* $S = x_1 < x_2 < \cdots$ *such that either*
1) *All* (x_α, x_β) *have the same color,*
2) (x_α,x_β) *and* (x_γ,x_δ) *have the same color if and only if* $\alpha=\gamma$,
3) (x_α,x_β) *and* (x_γ,x_δ) *have the same color if and only if* $\beta=\delta$, *or*
4) (x_α,x_β) *and* (x_γ,x_δ) *have the same color if and only if* $\alpha=\gamma$ *and* $\beta=\delta$,
where $\alpha<\beta$ *and* $\gamma<\delta$.

If a coloring has one of properties 1) through 4) when restricted to pairs from a set S, then S is *canonical*. For each positive integer n, let $[n] =$

Mathematics Subject Classification (1991): 05C35, 05C55

$\{1,\ldots,n\}$. By the Compactness Principle (see [5]), Theorem 1 implies that for any l, there exists an $N=N(l)$ such that every coloring of $\binom{[N]}{2}$ gives rise to a canonical l-set $S\subseteq[N]$.

Definition. Let $S\subseteq\mathbb{N}$, and let c be a coloring of $\binom{S}{2}$ with arbitrarily many colors. Suppose $l,k>0$. An l-flash is an increasing sequence $x_0,\ldots,x_l$ with $c(x_o,x_1)=\cdots=c(x_{l-1},x_l)$. A k-rainbow is an increasing sequence $y_0,\ldots,y_k$ with $c(y_0,y_1),\ldots,c(y_{k-1},y_k)$ pairwise distinct.

For the purpose of this article, an element $(i,j)\in\binom{S}{2}$ is called an *edge*. By writing (i,j) we mean that $i<j$ and we call i the *initial endpoint*.

If we linearly order the vertices of K_n, then the finite version of Theorem 1 implies that for n sufficiently large, every coloring of $E(K_n)$ yields either an l-flash or a k-rainbow. The question we consider is to determine the threshold for n to guarantee this weaker condition. When $k=l$, the minimum n that guarantees a canonical k-set is called $er(k)$. The best known upper bound, due to Lefmann and Rödl [3], is $er(k)\leq 2^{ck^2\log k}$, where c is a constant. Unfortunately, this result is of no help to us, since the threshold for n that guarantees a k-flash or a k-rainbow is much smaller than this upper bound.

Definition. Let $f(l,k)$ be the minimum n with the property that every coloring $c\colon\binom{[n+1]}{2}\to\{1,2,\ldots\}$ yields either an l-flash or a k-rainbow.

Stated another way, $f(l,k)$ is the maximum n such that there exists a coloring of $\binom{[n]}{2}$ with no l-flash and no k-rainbow.

It is shown in [4] that $f(l,k)\geq l^{k-1}$ by constructing a coloring of $\binom{[n]}{2}$ $(n=l^{k-1})$ yielding neither an l-flash nor a k-rainbow. Furthermore, it is also shown that $f(l,k)\leq l^{k-1}$ whenever $l\leq 2$, or $k\leq 4$, or $l\geq(3k)^{2k}$. In [4] the following conjecture is made:

Conjecture. *For all $l,k>0$, $f(l,k)=l^{k-1}$.*

Although we are unable to prove this conjecture, we furnish additional evidence for its truth. In particular, we prove it asymptotically when $l/k^2\to\infty$. This follows from the following theorem.

Theorem 2. *For $k\geq 5$, $l\geq 3$,*

$$f(l,k)<\left(1+\frac{1}{\sqrt{l}}\right)^{k-4}l^{k-1}.$$

In particular, if $k=o(\sqrt{l}\,)$, then $f(l,k)\sim l^{k-1}$ as $l\to\infty$.

2. Proof of the upper bound

In this section we provide the proof of Theorem 2. For fixed l, we use induction on k.

Lemma 3. *Writing f_k for $f(l,k)$, the bound in Theorem 2 follows from the recurrence*

$$f_k \leq l f_{k-1} + (l-1) f_{k-2} + l(l-1) f_{k-3} + \cdots + l^{k-3}(l-1) f_1 \tag{2.1}$$

and from the initial values $f_j = l^{j-1}$ for $j=1,2,3,4$.

Proof. The k^{th} term of the sequence obtained by replacing the inequality in (2.1) with equality bounds f_k from above, thus we may assume that equality holds in (2.1).

We assume that $k \geq 6$, since the algebraic manipulations we perform on (2.1) require this. The recurrence for f_{k-1} is

$$f_{k-1} = l f_{k-2} + (l-1) f_{k-3} + l(l-1) f_{k-4} + \cdots + l^{k-4}(l-1) f_1.$$

Multiplying both sides by l gives

$$l f_{k-1} = l^2 f_{k-2} + l(l-1) f_{k-3} + l^2(l-1) f_{k-4} + \cdots + l^{k-3}(l-1) f_1. \tag{2.2}$$

Subtracting (2.2) from (2.1) yields

$$f_k = 2l f_{k-1} - (l^2 - l + 1) f_{k-2} \qquad \text{for } k \geq 6. \tag{2.3}$$

The roots of the characteristic equation of the recurrence in (2.3) are $l \pm \sqrt{l-1}$, and thus a general solution to (2.3) is

$$f_k = c_1 (l + \sqrt{l-1}\,)^k + c_2 (l - \sqrt{l-1}\,)^k \tag{2.4}$$

for some constants c_1 and c_2.

Calculating f_5 directly from (2.1) yields $f_5 = l^4 + 3l^2(l-1)$. Using this and $f_4 = l^3$ in (2.4) we obtain

$$c_1 = \frac{l^3 + 3l^2\sqrt{l-1}}{2(l+\sqrt{l-1}\,)^4} \qquad \text{and} \qquad c_2 = \frac{l^3 - 3l^2\sqrt{l-1}}{2(l-\sqrt{l-1}\,)^4}\,.$$

This gives

$$\begin{aligned} f_k &= \frac{l^3 + 3l^2\sqrt{l-1}}{2(l+\sqrt{l-1}\,)^4}\,(l+\sqrt{l-1}\,)^k + \frac{l^3 - 3l^2\sqrt{l-1}}{2(l-\sqrt{l-1}\,)^4}\,(l-\sqrt{l-1}\,)^k \\ &< \frac{l^3 + 3l^2\sqrt{l-1}}{2}\,(l+\sqrt{l}\,)^{k-4} + \frac{l^3 - 3l^2\sqrt{l-1}}{2}\,(l+\sqrt{l}\,)^{k-4} \\ &= (l+\sqrt{l}\,)^{k-4}\, l^3 = \left(1 + \frac{1}{\sqrt{l}}\right)^{k-4} l^{k-1}. \end{aligned}$$

∎

We now derive the recurrence in (2.1).

Writing N for the right hand side of (2.1), we will show that every coloring of $\binom{[N+1]}{2}$ yields an l-flash or a k-rainbow. Let c be such a coloring. For convenience, we will say that a subset S of $[N+1]$ spans an l- flash or a k-rainbow if c restricted to $\binom{S}{2}$ has an l-flash or a k-rainbow. For the rest of the proof we will assume that $[N+1]$ spans no l-flash, and show that it spans a k-rainbow. In our notation $[f_{k-1}+1]$ is the subset of $[N+1]$ consisting of the first $f_{k-1}+1$ points. Since $[N+1]$ spans no l-flash, by the induction hypothesis $[f_{k-1}+1]$ spans a $k-1$-rainbow $R=\langle x_0,\dots,x_{k-1}\rangle$. By symmetry, we may assume that $c(x_{i-1},x_i)=i$ for $i=1,\dots,k-1$.

For each i and $j=0,1,\dots,l-1$, let S_j^i consist of those points x of $[N+1]$ for which the longest flash in color i ending at x has length j. Since $[N+1]$ spans no l-flash, we have for each color i that $[N+1]=\cup_{j=0}^{l-1}S_j^i$.

Observe that no edge (x,y) with both endpoints in S_j^i has color i, since otherwise the j-flash in color i ending at x can be extended to a $j+1$-flash ending at y by the edge (x,y), contradicting $y\in S_j^i$.

If $|S_j^i|>f_{k-1}$ for some $j\in\{1,\dots,l-1\}$, then by the induction hypothesis S_j^i spans a $k-1$-rainbow. This rainbow contains no edge with color i and can therefore be extended backward from its initial point by an edge with color i to yield a k-rainbow. Thus we may assume that $|S_j^i|\le f_{k-1}$ for each i and all $1\le j\le l-1$.

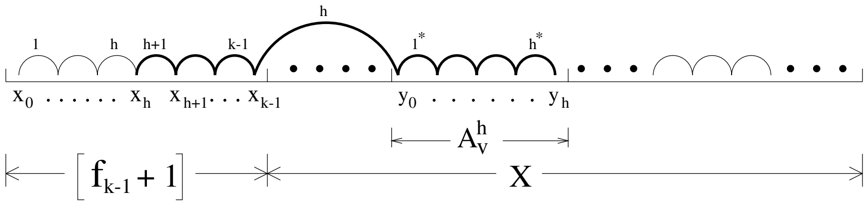

Fig. 1. A k-rainbow in $[N+1]$

Let

$$X=S_0^{k-1}-[f_{k-1}+1]=[N+1]-(\cup_{j=1}^{l-1}S_j^{k-1})-[f_{k-1}+1].$$

By definition, no edge with endpoints in X has color $k-1$. Moreover, by the above discussion, we may assume that $|\cup_{j=1}^{l-1}S_j^{k-1}|\le(l-1)f_{k-1}$. Notice also that $|[f_{k-1}+1]\cap(\cup_{j=1}^{l-1}S_j^{k-1})|\ge 1$ since $x_{k-1}\in[f_{k-1}+1]\cap(\cup_{j=1}^{l-1}S_j^{k-1})$. Hence we have $|X|\ge(N+1)-(l-1)f_{k-1}-(f_{k-1}+1)+1=N+1-lf_{k-1}>N-lf_{k-1}$.

The idea of the rest of the proof is to find an appropriate rainbow in X and concatenate it with part of the current rainbow $R=\langle x_0,\dots,x_{k-1}\rangle$

to obtain a k-rainbow (see Figure 1). If there is a point $x > x_{k-1}$ with $c(x_{k-1},x) > k-1$, then already we can extend the current rainbow using the new color. We may therefore assume that every edge with initial endpoint x_{k-1} has color less than k. In particular, $c(x_{k-1},x) < k$ for all $x \in X$. Also, by the definition of X, $c(x_{k-1},x) \neq k-1$ for all $x \in X$. Therefore we can partition the set of points $x \in X$ into $k-2$ sets depending on the color of the edge (x_{k-1},x). For $1 \le i \le k-2$, let

$$A^i = \{x \in X : c(x_{k-1},x) = i\}.$$

We have $X = \cup_{i=1}^{k-2} A^i$. Since $|X| > N - lf_{k-1} = \Sigma_{i=1}^{k-2} l^{k-2-i}(l-1)f_i$, the generalized pigeonhole principle implies that $|A^h| > l^{k-2-h}(l-1)f_h$ for some $1 \le h \le k-2$.

For each vector

$$\mathbf{v} = (t_h, t_{h+1}, \ldots, t_{k-2}) \in [1, l-1] \times [0, l-1]^{k-2-h} = \mathbf{I}$$

let $A^h_{\mathbf{v}} \subset A^h$ consist of those points y of A^h for which the maximum flash in color j ending at y has length t_j for each $j \in \{h, h+1, \ldots, k-2\}$. We have $A^h = \cup_{\mathbf{v} \in \mathbf{I}} A^h_{\mathbf{v}}$. By the pigeonhole principle, there is a vector $\mathbf{v} \in \mathbf{I}$ such that $|A^h_{\mathbf{v}}| > f_h$.

If an edge (x,y) with endpoints in $A^h_{\mathbf{v}}$ has color j, where $h \le j \le k-2$, then a t_j-flash in color j ending at x can be extended to a t_j+1-flash in color j ending at y, contradicting $y \in A^h_{\mathbf{v}}$. Hence each of the colors $h, \ldots, k-2$ is forbidden within $A^h_{\mathbf{v}}$. Recall that color $k-1$ is also forbidden in $A^h_{\mathbf{v}}$. By the induction hypothesis $A^h_{\mathbf{v}}$ contains an h-rainbow $\langle y_0, \ldots, y_h \rangle$ which avoids each of the colors $h, h+1, \ldots, k-1$. By concatenating this rainbow with part of R we obtain the k-rainbow $\langle x_h, \ldots, x_{k-1}, y_0, \ldots, y_h \rangle$ (see Figure 1). ∎

Remark. With more careful arguments, we can improve recurrence (2.1) slightly to $f_k \le lf_{k-1} + (l-1)f_{k-2} + l(l-1)f_{k-3} + \cdots + l^{k-3}(l-1)f_1 - (k-3)$.

Unfortunately, this improvement in the recurrence gives only a negligible improvement in the bound for $f(l,k)$ obtained in Theorem 2. When l is fixed and $k \to \infty$, the upper and lower bounds for $f(l,k)$ still differ by a factor exponential in k.

Acknowledgements. The authors thank H. Lefmann, V. Rödl, R. Thomas for helpful comments, and D. B. West for improving the presentation of the paper.

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