Small Odd Cycles in 4-Chromatic Graphs

Tao Jiang
DEPARTMENT OF MATHEMATICAL SCIENCES
MICHIGAN TECHNOLOGICAL UNIVERSITY
HOUGHTON, MICHIGAN 49931
E-mail: tjiang@mtu.edu

Received August 3, 1999

Abstract: It is shown that every 4-chromatic graph on n vertices contains an odd cycle of length less than $2\sqrt{n}+3$. This improves the previous bound given by Nilli [J Graph Theory 3 (1999), 145–147]. © 2001 John Wiley & Sons, Inc.
J Graph Theory 37: 115–117, 2001

Keywords: *odd cycle; chromatic number; girth*

By a theorem of Kierstead, Szemerédi, and Trotter [1], every 4-chromatic graph on n vertices contains an odd cycle of length at most $8\sqrt{n}$. On the other hand, there are infinitely many values of n for which there are 4-chromatic graphs on n vertices containing no odd cycle of length less than $\sqrt{2n}$. Constructions of such graphs appear in [4,5,2], where, in the last article, the authors raise the natural problem of determining the supremum of the reals c such that there are infinitely many values of n for which there are 4-chromatic graphs on n vertices with no odd cycles of length less than $c\sqrt{n}$. By the above results, it is known that this supremum is between $\sqrt{2}$ and 8. Recently, Nilli [3] improved the upper bound by showing that every 4-chromatic graph on n vertices contains an odd cycle of length less than $\sqrt{8n}$. Hence, the supremum is now between $\sqrt{2}$ and $\sqrt{8}$. In this

1990 Mathematics Subject Classification 05C35, 05C55

note, we further improve the upper bound to 2. Our method is based on the one Nilli uses.

Theorem 1. *Every* 4-*chromatic graph on n vertices contains an odd cycle of length less than* $2\sqrt{n}+3$.

Proof. Let G be a 4-chromatic graph on n vertices. Suppose the shortest odd cycle of G has length $2k+3$, we prove that $n > k^2$, from which we have $k < \sqrt{n}$, and the theorem follows.

Let A and B be two disjoint independent sets in G for which the cardinality $|A \cup B|$ is maximum. Since G is 4-chromatic, $G - A$ contains an odd cycle. Let C be a shortest odd cycle in $G - A$; clearly it is an induced cycle. By our assumption, C has length at least $2k+3$. Let v be a vertex on C. For each integer $i \geq 0$, define $V_i = \{u \in V(G) : d(u, v) = i\}$, where $d(u, v)$ is the distance between u and v in G. In particular, $V_0 = \{v\}$. Since G has no odd cycle of length at most $2k+1$, V_i is an independent set for each $i \leq k$. For each i, $1 \leq i \leq k$, define H_i to be the subgraph of G induced by

$$U_i = (\cup_{0 \leq j < i} V_j) \cup (V_i \cap B) \cup (\cup_{j>i}(V_j \cap (A \cup B))) \tag{1}$$

We claim that H_i is bipartite. Let $S = (\cup_{0 \leq j < i} V_j)$ and $T = \cup_{j>i}(V_j \cap (A \cup B))$. Note that there is no edge between S and T, since vertices in S are at distance at most $i-1$ from v, whereas vertices in T are at distance at least $i+1$ from v. Since V_j is an independent set for each $j \leq i$, the subgraph H_i' induced by $S \cup (V_i \cap B)$ is bipartite with a bipartition X_1, Y_1, such that $(V_i \cap B) \subseteq Y_1$. On the other hand, the subgraph induced by T is also bipartite with a bipartition $X_2 = T \cap A, Y_2 = T \cap B$. Since there is no edge between S and T, and B is an independent set, the only possible edges between H_i' and H_i'' are between $V_i \cap B$ and $T \cap A = X_2$. Since $V_i \cap B \subseteq Y_1$ by our assumption, H_i is bipartite with a bipartition $X_1 \cup X_2, Y_1 \cup Y_2$. It follows that U_i is the union of two disjoint independent sets. By our choice of A, B, we have $|A \cup B| \geq |U_i|$. In particular, $|A \cup B - U_i| \geq |U_i - A \cup B|$. Note that U contains $2i-1$ vertices on C that are within distance $i-1$ from v. These vertices induce a path in $G - A$. Since B is an independent set, it contains at most i of those vertices, hence at least $i-1$ of those vertices do not belong to $A \cup B$. This shows that $|U_i - A \cup B| \geq i-1$. On the other hand, it is easy to see that $A \cup B - U_i = V_i \cap A$. It follows that $|V_i \cap A| = |A \cup B - U_i| \geq |U_i - A \cup B| \geq i-1$. Hence, we have

$$|A| \geq \Sigma_{i=1}^{k} |V_i \cap A| \geq \Sigma_{i=1}^{k} (i-1) = k(k-1)/2 \tag{2}$$

By symmetry, we have $|B| \geq k(k-1)/2$. Observe also that at least $k+2$ vertices on C do not lie in B, and hence not in $A \cup B$. Therefore,

$$n \geq |A| + |B| + (k+2) = k^2 + 2 > k^2 \tag{3}$$

This completes our proof. ∎

References

[1] H. Kierstead, E. Szemerédi, and W. T. Trotter, On coloring graphs with locally small chromatic number, Combinatorica 4 (1984), 183–185.

[2] N. V. Ngoc and Zs. Tuza, 4-chromatic graphs with large odd girth, Discrete Math 138 (1995), 387–392.

[3] A. Nilli, Short odd cycles in 4-chromatic graphs, J Graph Theory 31 (1999), 145–147.

[4] H. Sachs and M. Stiebitz, Constructive methods in the theory of color-critical graphs, Discrete Math 74 (1989), 201–226.

[5] D. A. Youngs, 4-chromatic projective graphs, J Graph Theory 21 (1996), 219–227.