Ramsey numbers of some bipartite graphs versus complete graphs

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Abstract. The Ramsey number $r(H, K_n)$ is the smallest positive integer N such that every graph of order N contains either a copy of H or an independent set of size n. The Turán number $ex(m, H)$ is the maximum number of edges in a graph of order m not containing a copy of H. We prove the following two results:

(1) Let H be a graph obtained from a tree F of order t by adding a new vertex w and joining w to each vertex of F by a path of length k such that any two of these paths share only w. Then $r(H, K_n) \leq c_{k,t} \frac{n^{1+1/k}}{\ln^{1/k} n}$, where $c_{k,t}$ is a constant depending only on k and t. This generalizes some results in [11], [13], and [16].

(2) Let H be a bipartite graph with $ex(m, H) = O(m^\gamma)$, where $1 < \gamma < 2$. Then $r(H, K_n) \leq c_H \left(\frac{n}{\ln n}\right)^{1/(2-\gamma)}$, where c_H is a constant depending only on H. This generalizes a result in [4].

Key words. Ramsey number, independence number

1. Introduction

Let H, F be graphs without isolated vertices. The Ramsey number $r(H, F)$ is the smallest positive integer such that for each graph G on N vertices, either G contains H as a subgraph or its complement $\overline{G}$ contains F as a subgraph. For the purpose of this paper, we are only concerned with $r(H, K_n)$, which can be interpreted as the smallest positive integer N such that each graph on N vertices either contains H as a subgraph or has independence number at least n. For $m \geq 3$, the lower bound $r(K_m, K_n) \geq c_m \left(\frac{n}{\ln n}\right)^{(m+1)/2}$ was established by Spencer [14], where $\ln n$ is the natural logarithm function. For $m = 3$, Kim [9] obtained the asymptotically sharp lower bound $r(K_3, K_n) \geq cn^2/\ln n$. In general, Erdős et al. [5] and later Krivelevich [10] showed that if H is a graph with p vertices and q edges, then $r(H, K_n) \geq (c - o(1)) \left(\frac{n}{\ln n}\right)^{(q-1)/(p-2)}$, as $n \to \infty$. Li and Zang [12] generalized this lower bound by replacing K_n with a sequence of dense graphs.

For the upper bounds, Erdős et al. proved that $r(C_m, K_n) \leq c(m)n^{1+1/k}$, where $k = \lceil m/2 \rceil - 1$ and $c(m)$ is a constant depending on m. This bound was improved by Caro et

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al [4] for even m to $r(C_{2k}, K_n) \leq c(k)\,(n/\ln n)^{k/(k-1)}$. Sudakov [16] and independently Li and Zang [13] obtained a similar improvement for odd m, showing that $r(C_{2k+1}, K_n) \leq c(k) n^{1+1/k}/\ln^{1/k} n$. Li and Rousseau [11] proved that if F is a tree with m edges then $r(K_1 + F, K_n) \leq (m + o(1))n^2/\ln n$, where $K_1 + F$ is the graph obtained from F by adding a new vertex w and joining w to each vertex of F by an edge. Here we generalize these results by proving

Theorem 1. *Let n, k, t be positive integers. Let F be a tree with t vertices and H a graph obtained from F by adding a new vertex w and joining w to each vertex of F by a path of length k such that any two of these paths share only w. Then*

$$r(H, K_n) \leq c_{k,t} n^{1+1/k}/\ln^{1/k} n,$$

as $n \to \infty$, where $c_{k,t}$ is a constant depending only on k and t.

By taking $F = K_2$, Theorem 1 yields the above-mentioned upper bound on $r(C_{2k+1}, K_n)$. By taking F to be any tree and $k = 1$, Theorem 1 yields the correct order of magnitude in the above-mentioned upper bound on $r(K_1 + F, K_n)$. So, Theorem 1 directly generalizes the result in [16] and [13], but it only generalizes the result in [11] in a loose sense.

2. Proof of Theorem 1

In this section we prove Theorem 1. We follow the approach of [16] and [13]. Another key ingredient of our proof is a simple but useful lemma (Lemma 2) which potentially can be applied elsewhere.

We will assume whenever it is needed that n is sufficiently large. To keep the presentation concise we often choose multiplicative constants out of convenience, thus they are not optimal. We make no attempt to optimize these constants. We first need the following well-known bound ([3], Chapter 12, Lemma 15). See also [16].

Proposition 1. ([3]) *Let G be a graph on n vertices with average degree at most d and let h be the number of triangles in G. Then G contains an independent set of order at least*

$$0.1\frac{n}{d}\left(\ln d - 1/2\ln(h/n)\right).$$

It is well known that the independence number $\alpha(G)$ of a graph G with n vertices and maximum degree d is lower bounded by $\Omega(n/d)$. Proposition 1 can be applied to locally sparse graphs to give an improved lower bound of $\Omega(n \ln d/d)$, where loosely speaking a graph G is locally sparse if on average the neighborhood of a vertex spans $o(d^2)$ edges. See [1] for a detailed study of locally sparse graphs. Also, see [8] for an application to coding theory.

The following simple fact is frequently used in the literature. We include a proof for completeness.

Lemma 1. *Let T be a tree with t edges. If L is a n-vertex graph that does not contain T as a subgraph then $e(L) \leq tn(L)$.*

Proof. Suppose otherwise that $e(L) > tn(L)$. Let L' denote a smallest subgraph of L satisfying $e(L') > tn(L')$. Then the minimality of L' implies that L' has minimum degree at least t. It is well-known that every graph with minimum degree at least t contains every tree on t edges. Thus, L' contains T, contradicting L not containing T. ■

Proposition 1 and Lemma 1 readily imply the following.

Corollary 1. *Let H be a graph specified in Theorem 1. Suppose G is a graph on n vertices with maximum degree d which does not contain H as a subgraph. Then the independence number $\alpha(G)$ of G satisfies*

$$\alpha(G) \geq 0.05\frac{n}{d}(\ln d - \ln kt).$$

Proof. Let v be any vertex in G. Consider the subgraph L_v of G induced by the neighborhood $N(v)$ of v. Note that $n(L_v) = d(v) \leq d$. Since G does not contain H as a subgraph, L_v does not contain $F = H - w$. By our definition of H, F is a tree on $t + t(k-1) = kt$ vertices (and $kt - 1$ edges). By Lemma 1, $e(L_v) \leq kt \cdot n(L_v) \leq ktd$.

Now, if we sum up $e(L_v)$ over all vertices v, we would count each triangle in G three times, since in a triangle abc, we can designate any one of a, b, c as w. Hence, the total number h of triangles in G satisfies $h = \sum_v e(L_v)/3 \leq ktdn/3$. It follows from Proposition 1 that

$$\alpha(G) \geq 0.1\frac{n}{d}(\ln d - 1/2\ln(h/n)) \geq 0.1\frac{n}{d}(\ln d - 1/2\ln(ktd)) \geq 0.05\frac{n}{d}(\ln d - \ln kt).$$

■

The following lemma is key to our proof of Theorem 1. It generalizes the classical bound of Wei and independently of Caro on the independence number in terms of vertex degrees. A weaker version of the lemma was first proved by Haxell, Loh and Mubayi [7], followed by the first author of this paper. Sudakov [15] observed and proved the lemma in its current form. Even though the weaker version would suffice for our purpose, we present Sudakov's version because it is stronger (best possible) and the proof is short and elegant.

Lemma 2. ([15]) *Let G be an n-vertex graph together with a proper vertex-coloring c. For each vertex v, let $d_c(v)$ denote the number of different colors appearing on the neighbors of v. Let $D = \max_{v \in V(G)} d_c(v)$. Then*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + d_c(v)} \geq \frac{n}{D+1}.$$

Proof. Suppose the colors used by c are $1, \ldots, p$. Consider a random permutation σ of $\{1, \ldots, p\}$. Let X denote the set of vertices x whose color precedes the colors of all of its neighbors in σ. Let $E(|X|)$ denote the expected size of X. For each $v \in V(G)$, the probability of v belonging to X is precisely $\frac{1}{1+d_c(v)}$. Using indicator random variables and linearity of expectation, we have $E(|X|) = \sum_{v \in V(G)} \frac{1}{1+d_c(v)}$. So, there exists at least one permutation of $\{1, \ldots, p\}$ for which the corresponding X has size at least $\sum_{v \in V(G)} \frac{1}{1+d_c(v)}$. Fix such an X. Suppose X contains two adjacent vertices u and v. Then by the definition of X we must have both $c(u) < c(v)$ and $c(v) < c(u)$, which is impossible. So X is an independent set in G of size at least $\sum_{v \in V(G)} \frac{1}{1+d_c(v)}$. This proves the first inequality in the lemma. The second inequality is trivial. ■

If we take c to be the trivial coloring that gives each vertex of G a distinct color then Lemma 2 yields $\alpha(G) \geq \frac{1}{1+\Delta(G)}$, which is a well-known lower bound on independence number. If we take c to be a proper coloring of G using $\chi(G)$ colors, then trivially $D \leq \chi(G) - 1$ and Lemma 2 yields $\alpha(G) \geq \frac{1}{\chi(G)}$, which is another trivial lower bound on the independence number. In general, we believe that Lemma 2 is of independent interest, and that it should find applications elsewhere.

Given a graph L together with a vertex coloring c, a subgraph L' in it is *polychromatic* if all the vertices of L' have different colors.

Lemma 3. *Let T be a tree on t vertices. Let L be a graph together with a proper vertex-coloring c. Suppose that L contains no polychromatic copy of T. Then $\alpha(L) \geq 2n(L)/t(t+1)$.*

Proof. We use induction on t. For the basis step, let $t = 1$. Then $T = K_2$. If L contains no polychromatic K_2, then L contains no edge and $\alpha(L) = n(L)$. So the claim holds. For the induction step, let $t \geq 2$ and suppose the claim holds for all trees with fewer than t edges. Let T be a tree on t edges. Let z be a leaf of T and y its unique neighbor in T. Let $T' = T - y$. For each vertex x in L let $d_c(x)$ denote the number of different colors appearing on its neighbors. Let $S_1 = \{x \in V(L) : d_c(x) \geq t\}$ and $S_2 = \{x \in V(L) : d_c(x) \leq t-1\}$.

Suppose first that $L[S_1]$ contains a polychromatic copy T^* of T'. Let y^* denote the image of y in T^*. Since $y^* \in S_1$, $d_c(y^*) \geq t$. Since only $t-1$ colors are used on vertices of T^*, y^* has a neighbor z^* in L such that $c(z^*)$ is not used in T^*. In particular $z^* \notin V(T^*)$. Now we can extend T^* to a polychromatic copy of T in L via y^*z^*, contradicting L not containing a polychromatic copy of T. Hence $L[S_1]$ contains no polychromatic copy of T'. By the induction hypothesis, $\alpha(L[S_1]) \geq 2n(S_1)/(t-1)t$.

Now, since $d_c(x) \leq t-1$ for all x in $L[S_2]$, by Lemma 2, we have $\alpha(L[S_2]) \geq n(S_2)/t$. Let $n = n(L), n_1 = |S_1|, n_2 = |S_2|$. Then $n = n_1 + n_2$ and the discussions above show that $\alpha(L) \geq \max\{\frac{2n_1}{t(t-1)}, \frac{n_2}{t}\}$. It is easy to check that $\max\{\frac{2n_1}{t(t-1)}, \frac{n_2}{t}\}$ is minimized when $n_1 = \frac{t-1}{t+1}n$ and $n_2 = \frac{2}{t+1}n$ with minimum value $\frac{2n}{t(t+1)}$. Hence, $\alpha(L) \geq \frac{2n}{t(t+1)}$, completing the proof. ■

Lemma 4. *Let H be defined as in Theorem 1. For all $i \in [k]$ the following holds: if G is a graph not containing H as a subgraph and v is a vertex in G and $S_i(v)$ is a set of vertices at distance i from v in G then $\alpha(G[S_i(v)]) \geq |S_i(v)|/8(kt)^{2i-1}$.*

Proof. We use induction on i. For the basis step, let $i = 1$. Let G be any graph not containing H as a subgraph, v a vertex in G, and $S_1(v)$ a set of neighbors of v. Since G does not contain H as a subgraph, $G_1 = G[S_1(v)]$ does not contain $F = H - w$. By our definition of H, F is a tree on $t + t(k-1) = kt$ vertices. By Lemma 1, $e(G_1) \leq kt \cdot n(G_1)$. So, G_1 has average degree at most $2kt$. At most half of the vertices in G_1 can have degree at least $4kt$. By removing those vertices, we get an induced subgraph G_1' of G_1 with at least $n(G_1)/2$ vertices and maximum degree at most $4kt - 1$. So $\alpha(G_1) \geq \alpha(G_1') \geq \frac{n(G_1')}{1+\Delta(G_1')} \geq \frac{n(G_1)}{8kt}$. Thus, the lemma holds for $i = 1$.

For the induction step, let $j \geq 2$ and suppose the claim holds for all $i < j$, we prove the claim for $i = j$. Let G be a graph not containing H as a subgraph, v a vertex in G, and $S_j(v)$ a set of vertices at distance j from v in G. For each vertex u in $S_j(v)$, let P_u be a shortest v,u-path. The union of P_u over all u in $S_j(v)$ contains a tree T rooted at v containing $S_j(v)$ such that the distance from v to any vertex x in T is also the distance between v and x in G. (Note that T is just a subtree of the usual breath first search tree from v.) Let $x_1, \ldots, x_p$ denote the children of v in T. For each $i \in [p]$ let T_i denote the subtree of T rooted at x_i and let A_i denote the set of leaves of T_i. Clearly, $A_1, \dots, A_p$ partition $S_j(v)$.

For each $i \in [p]$, let B_i be a maximum independent set in $G[A_i]$. Since A_i is a set of vertices at distance $j-1$ from x_i, by induction hypothesis we have $|B_i| = \alpha(G[A_i]) \geq |A_i|/8(kt)^{2j-3}$.

Let $L = G[\bigcup_{i=1}^{p} B_i]$. Note that $n(L) = \sum_{i=1}^{p} |B_i| \geq |S_j(v)|/8(kt)^{2j-3}$. Let c be a proper coloring of L with $B_1, \dots, B_p$ being its color classes. Recall that H is formed by joining a vertex w to vertices of a tree F via paths of length k such that every two of these paths share only w. Let H_j denote the subgraph of H obtained by deleting w and vertices at distance at most $j-1$ from w in H.

Claim 1. *L contains no polychromatic copy of H_j.*

Proof of Claim 1. Suppose otherwise that L contains a polychromatic copy H' of H_j. For each $y \in V(H')$ let Q_y be the unique path in T of length j from y to the root v. If y lies in B_i then Q_y lies strictly inside $T_i \cup vx_i$. By our assumption, vertices of H' all lie in different B_i's. So every two of the Q_y's share only v. Furthermore, these Q_y's all have length j since $V(L) \subseteq S_j(v)$. Now the union of these paths and H' clearly contains a copy of H, contradicting G not containing H as a subgraph. ■

Now, since L contains no polychromatic copy of H_j, and H_j is a tree on at most kt vertices, by Lemma 3, $\alpha(L) \geq 2n(L)/kt(kt-1) \geq n(L)/(kt)^2 \geq \frac{1}{8(kt)^{2j-3}}|S_j(v)|/(kt)^2 \geq |S_j(v)|/8(kt)^{2j-1}$. This completes the induction step and the proof. ■

With all these preparatory lemmas, we are ready to prove Theorem 1. Our proof follows closely that of [16]. As usual, given a graph G, a vertex v in G and a nonnegative integer i, we use $N_i(v)$ to denote the set of all vertices at distance i from v.

Proof of Theorem 1: Let $c = c_{k,t} = 40(kt)^{2k-1}$. Let G be a graph on $m = (cn)^{1+1/k}/\ln^{\frac{1}{k}} n$ vertices that does not contain H as a subgraph. We prove that $\alpha(G) \geq n$. Let $d = c^{\frac{1}{k}} n^{1/k} \ln^{1-1/k} n$. We start with $G' = G$ and $I = \emptyset$ and as long as G' has a vertex of degree at least d we perform the following iterative procedure. Pick a vertex $v \in V(G')$ with degree at least d.

If $N_k(v)$ in G' has size at least cn, then by Lemma 4, $\alpha(G') \geq cn/8(kt)^{2k-1} \geq n$ by our choice of c and hence we are done. So we may assume that $|N_k(v)| \leq cn$. Since $|N_1(v)|/|N_0(v)| = |N_1(v)| \geq d$, there exists an index $i, 1 \leq i \leq k-1$ such that

$$\frac{|N_{i+1}(v)|}{|N_i(v)|} \leq \left(\frac{cn}{d}\right)^{\frac{1}{k-1}} = \frac{c^{\frac{1}{k}} n^{\frac{1}{k}}}{\ln^{\frac{1}{k}} n} = \beta. \tag{1}$$

Pick the smallest i satisfying (1). By Lemma 4, $G'[N_i(v)]$ has an independent set I' of size at least $|N_i(v)|/8(kt)^{2i-1} \geq 5|N_i(v)|/c$. Let $I = I \cup I'$ and remove all vertices in $N_{i-1}(v), N_i(v)$ and $N_{i+1}(v)$ from G'. The number of vertices we have removed is

$$|N_{i-1}(v)| + |N_i(v)| + |N_{i+1}(v)| \leq \left(\frac{1}{\beta} + 1 + \beta\right)|N_i(v)| \leq 2\beta|N_i(v)| \leq \frac{2c\beta}{5}|I'|, \tag{2}$$

and they contain all the neighbors of vertices in I'. Thus, during the whole process I stays as an independent set. Also, by (2) the ratio between the total number of vertices which we remove and the order of I is at most $\frac{2c\beta}{5}$.

Consider the final G' at the end of the process. Suppose first that there are at least $m/2$ vertices that remain. Since the process ends before we run out of vertices, it must be that no remaining vertex has degree at least d. By Corollary 1, using also that $c^{\frac{1}{k}} \geq kt$ and $c \geq 40k$, we have

$$\begin{aligned}\alpha(G') &\geq 0.05\frac{m/2}{d}(\ln d - \ln kt) = \frac{c^{1+\frac{1}{k}}n^{1+\frac{1}{k}}}{40\ln^{\frac{1}{k}} n} \cdot \frac{1}{c^{\frac{1}{k}}n^{\frac{1}{k}}\ln^{1-\frac{1}{k}} n} \cdot \ln\frac{c^{\frac{1}{k}}n^{\frac{1}{k}}\ln^{1-\frac{1}{k}} n}{kt}\\ &\geq \frac{cn}{40\ln n}\ln n^{\frac{1}{k}} = \frac{cn}{40k} \geq n.\end{aligned}$$

Since G' is an induced subgraph of G, we have $\alpha(G) \geq n$ as well and we are done.

Next, assume that fewer than $m/2$ vertices remain in G', which means at least $m/2$ vertices have been removed in the process. Then by earlier discussions we have

$$\alpha(G) \geq |I| \geq \frac{m/2}{2c\beta/5} = \frac{5}{4} \cdot \frac{c^{1+1/k}n^{1+1/k}}{\ln^{1/k} n} \cdot \frac{1}{c} \cdot \frac{\ln^{1/k} n}{c^{1/k}n^{1/k}} = \frac{5n}{4} > n.$$

This completes our proof of Theorem 1.

3. Ramsey numbers of general bipartite graphs versus complete graphs

In this short section, we give a general bound on $r(H, K_n)$ when H is bipartite. In [4], Caro et al. proved the following

Theorem 2. ([4]) *Suppose H is a subgraph of $K_1 + T$ where T is a tree, and $ex(m, H) = O(m^\gamma)$ for some γ satisfying $1 < \gamma < 2$ (of necessity H is bipartite). Then there exists a positive constant c_H such that $r(H, K_n) \leq c_H(\frac{n}{\ln n})^{\frac{1}{2-\gamma}}$.*

Here, we generalize their result so that it applies to all bipartite graphs H instead of just bipartite graphs that are subgraphs of $K_1 + T$ for some tree T.

Theorem 3. *Let H be a bipartite graph with $ex(m, H) = O(m^\gamma)$, where $1 < \gamma < 2$. Then there is a positive constant c_H depending only on H such that $r(H, K_n) \leq c_H \left(\frac{n}{\ln n}\right)^{\frac{1}{2-\gamma}}$.*

Proof. By our assumption, there exists an absolute constant a such that $ex(m, H) \leq am^\gamma$ for all m. Without loss of generality we may assume $a \geq 1$. Let $N = c_H(\frac{n}{\ln n})^{\frac{1}{2-\gamma}}$, where c_H is positive constant to be specified later. Let G be a graph on N vertices that does not contain H as a subgraph, we show that $\alpha(G) \geq n$.

Since G does not contain H as a subgraph, we have $e(G) \leq aN^\gamma$. So the average degree of G is at most $2aN^{\gamma-1}$. At most half of the vertices can have degree exceeding $4aN^{\gamma-1}$. By removing those vertices from G we get an induced subgraph G' of G with $n(G') \geq N/2$ and maximum degree $D = \Delta(G') \leq 4aN^{\gamma-1}$. Let $m = n(G') \geq N/2$. We may assume that D is a growing function of N; otherwise $\alpha(G') \geq m/(D+1) \geq c'N$, for some constant c'. With $N = c_H(\frac{n}{\ln n})^{\frac{1}{2-\gamma}}$, we would already have $\alpha(G') \geq n$ by choosing c_H to be large enough.

For any $v \in V(G')$, its neighborhood $N_{G'}(v)$ in G' contains no copy of H. Hence, $N_{G'}(v)$ spans at most $a[d_{G'}(v)]^\gamma \leq aD^\gamma$ edges. Hence the number h of triangles in G' is at most

$\frac{1}{3}m(aD^\gamma) < amD^\gamma$. By Proposition 1,

$$\begin{aligned}\alpha(G) \geq \alpha(G') &\geq \frac{m}{10D}\left(\ln D - \frac{1}{2}\ln(h/m)\right) \geq \frac{m}{20D}(2\ln D - \ln aD^\gamma)\\ &\geq c'N \cdot \frac{\ln D}{D} \quad \text{(for some positive constant } c')\\ &\geq c'N \cdot \frac{\ln 4aN^{\gamma-1}}{4aN^{\gamma-1}} \quad \text{(since } \tfrac{\ln D}{D} \text{ decreases in } D \text{ for } D \geq e)\\ &\geq c''N^{2-\gamma}\ln N \quad \text{(for some positive constant } c'').\end{aligned}$$

With $N = c_H(\frac{n}{\ln n})^{\frac{1}{2-\gamma}}$, one can ensure $c''N^{2-\gamma}\ln N \geq n$ by choosing c_H to be large enough. This completes the proof. ■

A careful reader may notice that in the proof above, the subgraph induced by the neighborhood of a vertex v in fact should not contain any subgraph of H of the form $H - x$, where $x \in V(H)$. So one might expect to get a better bound. However, in this case, such improvement would not affect the asymtotics.

As shown in [4], since $ex(m, C_{2k}) = O(m^{1+1/k})$, Theorem 3 immediately implies $r(C_{2k}, K_n) \leq c(k)(\frac{n}{\ln n})^{\frac{k}{k-1}}$ for some constant $c(k)$ depending on k, as mentioned in the introduction. Since $ex(m, K_{p,q}) = O(m^{2-1/p})$, where $p \leq q$, Theorem 3 also yields

Corollary 2. *Let p, q, n be positive integers where $p \leq q$. There exists a constant $c_{p,q}$ such that $r(K_{p,q}, K_n) \leq c_{p,q}(\frac{n}{\ln n})^p$.*

Alon, Krivelevich and Sudakov [2] proved that if H is a bipartite graph in which vertices in one partite set all have degree at most p then $ex(m, H) = O(m^{2-1/p})$. By Theorem 3 we have

Corollary 3. *Let p be a positive integer. Let H be a bipartite graph with a bipartition (X, Y) such that vertices in X all have degree at most p. Then there exists a positive constant c_H depending on H such that $r(H, K_n) \leq c_H(\frac{n}{\ln n})^p$.*

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