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NOTE

Anti-Ramsey Numbers of Subdivided Graphs

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Given a positive integer n and a family $\mathscr{F}$ of graphs, the *anti-Ramsey number* $f(n, \mathscr{F})$ is the maximum number of colors in an edge-coloring of K_n such that no subgraph of K_n belonging to $\mathscr{F}$ has distinct colors on its edges. The *Turán number* $ex(n, \mathscr{F})$ is the maximum number of edges of an n-vertex graph that does not contain a member of $\mathscr{F}$ as a subgraph. P. Erdős *et al.* (1975, *in* Colloq. Math. Soc. Janos Bolyai, Vol. 10, pp. 633–643, North-Holland, Amsterdam) showed for all graphs H that $f(n, H) - ex(n, \mathscr{H}) = o(n^2)$, where $\mathscr{H} = \{H - e : e \in E(H)\}$. We strengthen their result for the class of graphs in which each edge is incident to a vertex of degree two. We show that $f(n, H) - ex(n, \mathscr{H}) = O(n)$ when H belongs to this class. This follows from a new upper bound on $f(n, H)$ that we prove for all graphs H and asymptotically determines $f(n, H)$ for certain graphs H.

1. INTRODUCTION

We consider only nonempty simple graphs. A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. Given a positive integer n and a family $\mathscr{F}$ of graphs, the *anti-Ramsey number* $f(n, \mathscr{F})$ is the maximum number of colors in a coloring of $E(K_n)$ that has no rainbow copy of any graph in $\mathscr{F}$. For the purpose of this note, we call a coloring that does not contain a rainbow copy of any graph in $\mathscr{F}$ a $\mathscr{F}$-free coloring.

Anti-Ramsey numbers were introduced by Erdős *et al.* [4]. They showed that these are closely related to Turán numbers. The *Turán number* $ex(n, \mathscr{F})$ of $\mathscr{F}$ is the maximum number of edges of an n-vertex simple graph having no member of $\mathscr{F}$ as a subgraph. Given a coloring c of a host

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graph G, we define a *representing graph* of c to be a spanning subgraph L of G obtained by taking one edge of each color in c (where L may contain isolated vertices). Given a positive integer n and a graph H, clearly a representing graph of an H-free coloring of $E(K_n)$ does not contain H as a subgraph. Thus we have $f(n, H) \leqslant ex(n, H)$. Let $\mathscr{H} = \{H - e : e \in E(H)\}$. Let G be a subgraph of K_n with $ex(n, \mathscr{H})$ edges that does not contain any member of $\mathscr{H}$ as a subgraph. We can define an H-free coloring of $E(K_n)$ using at least $ex(n, \mathscr{H})$ colors by coloring the edges of G with distinct colors and then coloring the remaining edges (if any) in K_n with a new color. Hence, $f(n, H) \geqslant ex(n, \mathscr{H})$.

PROPOSITION 1.1. *Given a positive integer n and a graph H, we have*

$$ex(n, \mathscr{H}) \leqslant f(n, H) \leqslant ex(n, H),$$

where $\mathscr{H} = \{H - e : e \in E(H)\}$.

The lower and upper bound in Proposition 1.1 could differ even in the order of magnitude. For instance, when H is an odd cycle, $ex(n, H)$ is quadratic in n while $ex(n, \mathscr{H})$ is linear in n. In general, the upper bound $ex(n, H)$ is quite loose, and $f(n, H)$ is much closer to $ex(n, \mathscr{H})$. Erdős *et al.* [4] showed that $f(n, H) \leqslant ex(n, \mathscr{H}) + o(n^2)$ as $n \to \infty$. Thus we have

THEOREM A [4]. *$f(n, H) - ex(n, \mathscr{H}) = o(n^2)$, as $n \to \infty$.*

If $d = \min\{\chi(G) : G \in \mathscr{H}\} \geqslant 3$, then by an earlier result of Erdős and Simonovits [5], we have $ex(n, \mathscr{H}) = \frac{d-2}{d-1}\binom{n}{2} + o(n^2)$, and Theorem A yields $f(n, H) = \frac{d-2}{d-1}\binom{n}{2} + o(n^2)$. This determines $f(n, H)$ asymptotically. If $d \leqslant 2$, however, we have $ex(n, \mathscr{H}) = o(n^2)$, and Theorem A says little about $f(n, H)$. Erdős *et al.* [4] therefore proposed studying $f(n, H)$ for graphs H that contains an edge whose deletion leaves a bipartite subgraph, and they put forward two conjectures about $f(n, H)$ when H is a path or a cycle.

Simonovits and Sós [9] proved the conjecture for paths, showing for large n that $f(n, P_{2t+3+\varepsilon}) = tn - \binom{t+1}{2} + 1 + \varepsilon$, where $\varepsilon = 0$, 1 and P_k is a path on k vertices. Jiang and West [7] considered $f(n, T)$ when T is a general tree of a given size. For cycles, Erdős *et al.* [4] conjectured that for every fixed $k \geqslant 3$ $f(n, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1)$, and they obtained a C_k-free coloring of $E(K_n)$ using the conjectured number of colors. They noted that the conjecture holds for $k = 3$. Alon [1] proved the conjecture for $k \leqslant 4$ and proved that $f(n, C_k) \leqslant n(k-2) + \binom{k-1}{2}$ in general. Jiang and West [8] proved the conjecture for $k \leqslant 6$ and improved the general upper bound to $f(n, C_k) \leqslant n(\frac{k+1}{2} - \frac{2}{k-1}) - (k-2)$ for all k and to $f(n, C_k) \leqslant nk/2 - (k-2)$ when k is even. Axenovich and Jiang [3] initiated the study of the anti-Ramsey numbers for complete bipartite graphs. They showed for all $t \geqslant 3$ that $f(n, K_{2,t}) = \sqrt{t-2}\, n^{3/2} + O(n^{4/3})$ by proving that $f(n, K_{2,t}) - ex(n, K_{2,t-1}) = O(n)$.

Note that in the cases mentioned above when H is a path, a cycle, or a complete bipartite graph with one bipartite set of size 2, one has $f(n, H) - ex(n, \mathscr{H}) = O(n)$. In this note, we establish a more general fact that if H is a graph in which each edge is incident to a vertex of degree two then $f(n, H) - ex(\mathscr{H}) = O(n)$ always holds (which immediately implies the result obtained in [3]). In particular, this applies to graphs H obtained by subdividing each edge of any given graph G at least once. The claim follows from the following upper bound on $f(n, H)$ that holds for all (nonempty) graphs H.

Theorem 1.2. *Given a graph H, let $\mathscr{H}_2 = \{H - v : v \in V(H), d_H(v) = 2\}$. Suppose H has p vertices and q edges. For all positive integers n, we have*

$$f(n, H) \leqslant ex(n, \mathscr{H}_2) + bn,$$

where $b = \max\{2p-2, q-2\}$.

Now suppose H is a (nonempty) graph in which each edge is incident to a vertex of degree two. Let e be any edge in H. By our assumption, e is incident to a vertex v of degree two in H. Note that $H - e$ contains $H - v$ as a subgraph. This shows that every member of $\mathscr{H}$ contains a subgraph that is in $\mathscr{H}_2$. Thus we have $ex(n, \mathscr{H}_2) \leqslant ex(n, \mathscr{H})$. This observation together with Theorem 1.2 and Proposition 1.1 yields

Theorem 1.3. *Let H be a graph in which each edge is incident to a vertex of degree two. Suppose H has p vertices and q edges. Let $\mathscr{H} = \{H - e : e \in E(H)\}$ and $b = \max\{2p-2, q-2\}$. We have*

$$ex(n, \mathscr{H}) \leqslant f(n, H) \leqslant ex(n, \mathscr{H}) + bn.$$

Hence $f(n, H) - ex(n, \mathscr{H}) = O(n)$, as $n \to \infty$.

It is known that $ex(n, G)$ grows at least super linearly in n for any graph G which is not a forest. Hence Theorem 1.3 implies

Corollary 1.4. *If H is a graph containing at least two cycles in which each edge is incident to a vertex of degree two, then*

$$f(n, H) = ex(n, \mathscr{H})(1 + o(1)),$$

where $\mathscr{H} = \{H - e : e \in (H)\}$.

For the rest of the paper, we give a proof of Theorem 1.2. Given a graph G and a subset $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph of G induced by U. Given a vertex u in G, $N_G(u)$ denotes its neighborhood in G.

2. PROOF OF THEOREM 1.2

Let H be a given graph, and let $\mathscr{H}_2 = \{H - v : v \in V(H), d_H(V) = 2\}$. Suppose H has p vertices and q edges. Then each graph in $\mathscr{H}_2$ has $p-1$ vertices and $q-2$ edges. We introduce some notions for convenience. Given any graph $D \in \mathscr{H}_2$, by definition, $D = H - v$ for some vertex v of degree two in H. We use $a(D)$ and $b(D)$ to denote the two neighbors of v in H, and call them the two *ends* of D. Let $S(D) = \{a(D), b(D)\}$. A graph R is an $\mathscr{H}_2$*-string* of length k if the edges of R can be partitioned into k subgraphs $D_1, ..., D_k$ such that $D_i \in \mathscr{H}_2$ and $S(D_i) = \{u_i, u_{i+1}\}$ for all $i \in [k]$, where u_1, u_2, u_{k+1} are distinct vertices. Let $S(R) = \bigcup_{i=1}^{k} S(D_i) = \{u_1, u_2, ..., u_{k+1}\}$. Vertices u_1, u_{k+1} are the two *ends* of R. For $k \geqslant 2$, if in the above definition, $u_1, ..., u_k$ are distinct and $u_{k+1} = u_1$, then R is an $\mathscr{H}_2$*-ring* of length k.

LEMMA 2.1. *Let G be a graph on n vertices with more than $ex(n, \mathscr{H}_2) + (q-2)(n-1)$ edges. Then G contains an $\mathscr{H}_2$-ring.*

Proof. Recall that each graph in $\mathscr{H}_2$ has $q-2$ edges. Let $\mathscr{D}$ be a maximal collection of pairwise edge-disjoint subgraphs of G which belong to $\mathscr{H}_2$. Suppose $\mathscr{D}$ contains m members. By the maximality of $\mathscr{D}$, $G - E(\mathscr{D})$ contains no subgraphs that belong to $\mathscr{H}_2$. Hence we have $e(G - E(\mathscr{D})) \leqslant ex(n, \mathscr{H}_2)$. Thus, $e(G) \leqslant ex(n, \mathscr{H}_2) + m(q-2)$. Since $e(G) > ex(n, \mathscr{H}_2) + (q-2)(n-1)$, it follows that $m > n-1$. Now, construct a graph F with $V(F) = V(G)$ as follows. For each member D (which is a graph in $\mathscr{H}_2$) of $\mathscr{D}$, where $S(D) = \{u, v\}$, we include uv as an edge in F. Since $\mathscr{D}$ has m members, the resulting graph F is an n-vertex loopless multigraph with $m > n-1$ edges. Hence F contains a cycle C. The union of the members of $\mathscr{D}$ which correspond to the edges C forms an $\mathscr{H}_2$-ring in G. ∎

A graph T obtained from an $\mathscr{H}_2$-string R of length k by adding a new vertex x not in R and making it adjacent to the two ends of R is an $\mathscr{H}_2$ *string-tie* of length k. Note that H is an $\mathscr{H}_2$-string-tie of length 1.

LEMMA 2.2. *Let c be a coloring of $E(K_n)$ that contains a rainbow $\mathscr{H}_2$-string-tie. Then c contains a rainbow copy of H.*

Proof. Let T be a rainbow $\mathscr{H}_2$-string-tie in c of minimum length. Suppose T is obtained from an $\mathscr{H}_2$-string R of length k by adding a vertex x not in R and making it adjacent to the two ends of R. Suppose R is the edge-disjoint union of $D_1, ..., D_k$, where $D_i \in \mathscr{H}_2$, and $S(D_i) = \{u_i, u_{i+1}\}$ for all $i \in [k]$. If $k=1$ then T is a rainbow H. So we may assume $k \geqslant 2$. Let $T_1 = D_1 \cup xu_1$ and $T_2 = D_2 \cup \cdots \cup D_k \cup xu_{k+1}$. Since T is rainbow, the color $c(xu_2)$ cannot be used in both T_1 and T_2. Now xu_2 completes a rainbow $\mathscr{H}_2$-string-tie with either T_1 or T_2, which is shorter than T, a contradiction. ▮

LEMMA 2.3. *Suppose c is an H-free coloring of $E(K_n)$ and R is a rainbow $\mathscr{H}_2$-ring in c. Let $x \in V(K_n) - V(R)$. Suppose there exists $y \in S(R)$ such that the color $c(xy)$ is not used on the edges of R, then $c(xy') = c(xy)$ for all $y' \in S(R)$.*

Proof. Otherwise, suppose there exists $y' \in S(R)$ such that $c(xy') \neq c(xy)$. Vertices y and y' partition R into two $\mathscr{H}_2$-strings R_1, R_2 sharing y, y' as common ends. Since R is rainbow, one of R_1 and R_2 avoids the color $c(xy')$. Suppose R_1 does. Now $R_1 \cup \{xy, xy'\}$ is a rainbow $\mathscr{H}_2$-string-tie, and by Lemma 2.2, c contains a rainbow copy of H, a contradiction. ▮

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We use induction on n, with the claim holding trivially for small values of n. Let c be an H-free coloring of $E(K_n)$ using $f(n, H)$ colors. Let L be a representing graph of c. If L contains no $\mathscr{H}_2$-ring, then by Lemma 2.1, we have $f(n, H) = e(L) \leqslant ex(n, \mathscr{H}_2) + (q-2)(n-1) \leqslant ex(n, \mathscr{H}_2) + bn$, recalling that $b = \max\{2p-2, q-2\}$. So we may assume that L contains an $\mathscr{H}_2$-ring R of length k, where $k \geqslant 2$. Since c contains no rainbow H, by Lemma 2.2, L contains no $\mathscr{H}_2$-string-tie.

CLAIM 2.4. *The number of edges in $L[V(R)]$ that are incident to $S(R)$ is at most $(p-1)\,k$.*

Proof of Claim 2.4. Suppose R consists of $D_1, ..., D_k$, with $D_i \in \mathscr{H}_2$, and $S(D_i) = \{u_i, u_{i+1}\}$ for $i \in [k]$ (with indices taken modulo k). Let $v \in V(R)$. Suppose $N_L(v) \cap S(R) = \{u_{j_1}, u_{j_2}, ..., u_{j_m}\}$, where $j_1 < j_2 < \cdots < j_m$. For each $i \in [m]$, let $F_i = \bigcup_{l=j_i}^{j_{i+1}-1} D_l$ (with indices l taken modulo k). If $v \notin V(F_i)$ for some $i \in [m]$ then $F_i \cup \{vu_{j_i}, vu_{j_{i+1}}\}$ would form an $\mathscr{H}_2$-string-tie in L, a contradiction. Hence $v \in V(F_i)$ for all $i \in [m]$. So, in particular, v is contained in at least m of the D_i's. Hence we have

$$\begin{aligned}
&\#\text{ edges in } L[V(R)] \text{ incident to } S(R)\\
&\leqslant \sum_{v \in V(R)} \#\text{ edges in } L \text{ between } v \text{ and } S(R)\\
&\leqslant \sum_{v \in V(R)} |\{i \in [k] : v \in V(D_i)\}|\\
&= \sum_{i=1}^{k} |V(D_i)| = k(p-1).
\end{aligned}$$

Now, let $K = K_n$, and let $K' = K - \{u_1, \ldots, u_{k-1}\}$. Consider any color α that is used by c in K but not in K'. By the definition of L as a representing graph of c, there is an edge e of L such that $c(e) = \alpha$. Since α is not used in K', one of the endpoints of e must be in $\{u_1, \ldots, u_{k-1}\}$. Suppose $e = xu_i$, where $i \in [k-1]$. Suppose $x \notin V(R)$. Then xu_i does not lie in R and therefore $c(xu_i)$ is not used on the edges of R (recall that edges of L have distinct colors). By Lemma 2.3, we have $\alpha = c(xu_i) = c(xu_k)$, contradicting our assumption that α is not used in K' (note that $xu_k \in E(K')$). Hence $x \in V(R)$, and $\alpha = c(xu_i)$ is used on an edge of $L[V(R)]$ that is incident to $S(R)$. By Claim 2.4, there are at most $(p-1)/k$ such colors. Now, since K' is a complete graph of order $n-k+1$, and c restricted to K' is H-free, by induction hypothesis we have

$$\begin{aligned} f(n, H) - (p-1)\,k &\leqslant \#\text{ colors used by } c \text{ in } K' \\ &\leqslant ex(n-k+l, \mathscr{H}_2) + b(n-k+1). \end{aligned}$$

Hence, $f(n, H) \leqslant ex(n-k+1, \mathscr{H}_2) + b(n-k+1) + (p-1)\,k \leqslant ex(n, \mathscr{H}_2) + bn$, recalling that $b = \max\{2p-2, q-2\}$ and $k \geqslant 2$. This completes the proof of Theorem 1.2. ∎

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