Compact topological cliques in sparse graphs

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Abstract

Let ϵ be a real number such that $0 < \epsilon < 1$ and t a positive integer. Let n be a sufficiently large positive integer as a function of t and ϵ. We show that every n-vertex graph with at least $2^{7t^2} n^{1+\epsilon}$ edges contains a subdivision of K_t in which each edge of K_t is subdivided at most $\max\{2, \frac{10}{\epsilon}\ln\frac{1}{\epsilon}\} - 1$ times. This improves the main result in [12]. Some related problems are proposed.

1 Introduction

We consider only simple graphs in this paper unless otherwise specified. Generally speaking, in extremal graph theory we study how global parameters of a graph such as its edge density or chromatic number affect the existence of certain substructures. One of the most fundamental substructures is a subgraph. Given a graph H, how many edges do we need in an n-vertex graph G to force H to appear as a subgraph of G? Turán was one of the first to explore this question. The classic Turán number $ex(n, H)$ of a graph H is defined as the maximum number of edges in an n-vertex graph not containing H as a subgraph. Turán determined the exact value of $ex(n, K_r)$. The celebrated Erdős-Simonovits-Stone Theorem determined $ex(n, H)$ asymptotically for all non-bipartite H, showing that $ex(n, H) \sim (1 - \frac{1}{p-1})\binom{n}{2}$, where $p = \chi(H)$ is the chromatic number of p.

Another fundamental substructure of interest is the so-called *minor*. Given a graph H and another graph L, L is called a H-minor if H can be obtained from L by a series of edges contractions. If a graph G contains a subgraph that is an H-minor, then we can also say that G contains H as a minor. (In other words, G contains a subgraph that can be contracted into H.) A variant of the notion of a minor is that of a *topological minor* of *subdivision*. Subdividing an edge uv in a graph, as usual, means replacing the edge uv by path uwv through a new vertex w. Given a graph H and another graph L, we say that L is an H-subdivision (a subdivision

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of H) if L can be obtained from H by series of edge subdivisions. For example, any cycle C can be viewed as a C_3-subdivision, as it can be obtained from C_3 via edge-subdivisions. If a graph G contains a subdivision of H, we also say that G contains H as a topological minor. A famous example of use of the notion of subdivisions is the Kuratowski's Theorem which says a graph is planar if and only if it contains no subdivisions of K_5 or $K_{3,3}$.

Mader was one of the first to explore Turán type questions for (topological) minors. We have seen from the Erdős-Simonovits-Stone Theorem that in order to force any given non-bipartite H to appear as a subgraph in an n-vertex host graph, we need $\Omega(n^2)$ edges. Indeed, $K_{n/2,n/2}$ has $n^2/4$ edges and avoids any non-bipartite H as a subgraph. So, loosely speaking it takes a very dense host graph to force a particular H to appear as a subgraph. Contrasting this phenomenon for subgraphs, Mader showed that it only takes $O(n)$ edges in an n-vertex host graph to force a subdivision of H, as implied by the following theorem of his.

Theorem 1.1 (Mader [13]) *Given a positive integer t, there exists a positive constant c_t depending only on t such that every n-vertex graph G with at least $c_t n$ edges contains a subdivision of K_t.*

While Mader's theorem was proved in 1967, it was not until mid til late 1990s that Bollobás and Thomason [3] and independently Komlós and Szemerédi [10] determined the growth rate of the optimal constant c_t as a function of t, showing that Mader's Theorem holds for some $c_t \leq ct^2$ for some absolute constant c.

We see between Turán's theorem and Mader's theorem that there is a big gap between required edge-densities for forcing a graph H directly as a subgraph versus as a topological minor (subdivision), with the former much higher. What seems to make it easier to force a subdivision of H is that we allow as many subdivisions on an edge of H as needed and this gives us a lot of leeway. This is reminiscent of forcing cycles: it is much easier to force a cycle of length at least a given length k than to force a cycle of length at most k (for the former $O(n)$ edges suffice while for the latter $\Omega(n^{1+1/(k+1)})$ edges are required).

This brings up a natural question: what if we require that each edge in the original H is only allowed to be subdivided some bounded number p times? What edge density is needed in a host graph G to force such a more restricted subdivision of H? Formally, we introduce the following definition.

Definition 1.2 Let p be a positive integer. A subdivision L of a graph H is called a *p-subdivision* of H if in forming L each edge of H is subdivided at most $p-1$ times.

Note that a 1-subdivision of H is just H itself. We pose the following general question.

Question 1.3 *Fix a positive integer p and a graph H, how many edges does an n-vertex graph G need to have to force a p-subdivision of H?*

We may rephrase Question 1.3 in terms of Turán numbers as follows. Given a positive integer p and a graph H, let $H^{(\leq p)}$ denote the family of p-subdivisions of H (i.e. graphs obtainable from H by subdividing each edge of H at most $p-1$ times).

Problem 1.4 *Given positive integers p, n and a graph H, determine $ex(n, H^{(\leq p)})$.*

In this paper, we focus on the case where $H = K_t$, the complete graph, as it provides natural bounds for the general H. Another motivation for our work in this paper comes from the following work of Kostchoka and Pyber [12]. It is well-known that any n-vertex planar graph with more than $3n - 6$ edges contains a non-planar subgraph. However, if we want to force a non-planar subgraph of a bounded (constant) order then it is easy to see that cn edges are not sufficient no matter how large c is. Erdős [5] asked the following: Is it true that every n-vertex graph with at least $n^{1+\epsilon}$ edges contains a non-planar subgraph of order at most $c(\epsilon)$, where $c(\epsilon)$ is a constant depending only on t? Kostochka and Pyber [12] answered the question in the affirmative, proving the following more general result.

Theorem 1.5 (Kostocha and Pyber [12]) *Let ϵ be a positive real such that $0 < \epsilon < 1$. Let n, t be positive integers. Every n-vertex graph with at least $4^{t^2} n^{1+\epsilon}$ edges contains a subdivision of K_t on at most $7t^2 \ln t/\epsilon$ vertices.*

Since a subdivision of K_5 is non-planar, the special case of $k = 5$ of Theorem 1.5 answers Erdős' question in the affirmative. In proving Theorem 1.5, Kostochka and Pyber in effect proved

Theorem 1.6 [12] *Let ϵ be a positive real such that $0 < \epsilon < 1$. Let n, t be positive integers. Every n-vertex graph with at least $4^{t^2} n^{1+\epsilon}$ edges contains a p-subdivision of K_t with $p \leq 2(1 + \frac{2}{\epsilon}(1 + 2\ln t)) \leq \frac{14 \ln t}{\epsilon}$.*

Note that in Theorem 1.6, the bound on p varies with t. If we use Theorem 1.6 to derive a bound on $ex(n, K_t^{(\leq p)})$, then we only get $ex(n, K_t^{(\leq p)}) \leq 4^{t^2} n^{1+\frac{14 \ln t}{p}}$, which is not very useful since as t grows the exponent of n quickly exceeds 2 and thus trivializing the bound.

In this paper, we eliminate this dependency of the bound of p on t (Theorem 1.10). First we note that when $\epsilon \geq \frac{1}{2}$ the following result of Alon, Krivelevich and Sudakov [1] readily implies the existence of a 2-subdivision of K_t in any n-vertex graph with at least $\frac{t^2}{2} n^{1+\epsilon}$ edges.

Proposition 1.7 [1] *Let H be a bipartite graph with maximum degree r on one side. Then exists a constant $c_H > 0$, depending on H, such that $ex(n, H) \leq c_H n^{2-\frac{1}{r}}$.*

From the proof of Theorem 2.2 in [1], one can easily check that $c_H \leq n(H)$ when $r = 2$. Thus, we have

Corollary 1.8 *Let H be a bipartite graph with h vertices such that vertices in one partite set all have degree at most 2, then $ex(n, H) \leq \frac{h}{2} n^{3/2}$.*

Let $K_t^{(2)}$ denote the graph obtained from K_t by subdividing each edge exactly once. Observe that $K_t^{(2)}$ is a bipartite graph on $t + \binom{t}{2} < t^2$ vertices in which the vertices used to subdivide the edges form one part X and the branching vertices form the other part Y. Further, each vertex in X has degree two. By Corollary 1.7, we have

Proposition 1.9 *For each positive integer t, we have $ex(n, K_t^{(\leq 2)}) \leq ex(n, K_t^{(2)}) \leq \frac{t^2}{2} \cdot n^{3/2}$.*

Proposition 1.9 gives the correct order of magnitude for $ex(n, K_t^{(\leq 2)})$ and $ex(n, K_t^{(2)})$ because of the well-known fact that there are n-vertex graphs with $\Omega(n^{3/2})$ edges and no C_3 or C_4. Proposition 1.9 implies that for $\epsilon \geq \frac{1}{2}$ every n-vertex graph with at least $\frac{t^2}{2}n^{1+\epsilon}$ edges contains a 2-subdivision of K_t. Thus, for the rest of the paper, we restrict our attention to $\epsilon < 1/2$. Our main result is as follows.

Theorem 1.10 *Let t be a positive integer. Let $0 < \epsilon < 1/2$ be a real. Let G be an n-vertex graph with at least $2^{7t^2} \cdot n^{1+\epsilon}$ edges, where n is sufficiently large as a function of t and ϵ. Then G contains a $\frac{10}{\epsilon}\ln\frac{1}{\epsilon}$-subdivision of K_t.*

It is well-known that there are n-vertex graphs with $\Omega(n^{1+1/g})$ edges and girth at least g. Note that a p-subdivision of K_t, where $t \geq 3$, necessarily contains a cycle of length at most $3p$. Hence, there are n-vertex graphs with $\Omega(n^{1+1/(3p+1)})$ edges and no p-subdivision of K_t. This implies

Proposition 1.11 *For each real $0 < \epsilon < 1$ and positive integer $t \geq 3$, there are n-vertex graphs with $\Omega(n^{1+\epsilon})$ edges and no $\lfloor\frac{1}{3}(\lfloor\frac{1}{\epsilon}\rfloor - 1)\rfloor$-subdivision of K_t.*

Proposition 1.11 shows that Theorem 1.10 is not too far from being optimal. However, it is natural to ask whether one can get rid of the $\ln\frac{1}{\epsilon}$ factor in Theorem 1.10. Most importantly, compared to Theorem 1.5, the new bound on the number of times each edge of K_t is subdivided now depends only on ϵ and does not vary with t. This, together with Proposition 1.9, allows us to obtain the following bound on $ex(n, K_t^{(\leq p)})$. For natural reasons, we may assume $p \geq 2$ and $t \geq 3$.

Corollary 1.12 *Let p, t be fixed positive integers where $p \geq 2$ and $t \geq 3$. As a function of n, we have $ex(n, K_t^{(\leq p)}) = O(n^{1+\min\{\frac{10\ln p}{p}, 1/2\}})$ and $ex(n, K_t^{(\leq p)}) = \Omega(n^{1+1/(3p+1)})$.*

If one can somehow remove the $\ln\frac{1}{\epsilon}$ factor in Theorem 1.10, then one would improve the upper bound on $ex(n, K_t^{(\leq p)})$ to $O(n^{1+\frac{c}{p}})$ for some constant c. For the rest of the paper, we prove Theorem 1.10. We will combine ideas from [12], [8], as well as some new ideas. In our arguments, we will drop floors and ceilings whenever appropriate. For terms and notation not defined here, the reader is referred to [16].

2 Forcing compact subdivisions in graphs with no dense subgraphs of small radius

To prove Theorem 1.10, we first prove in this section that if a graph is reasonably dense itself but contains no dense subgraph of small radius, then we can find a compact subdivision of K_t. The following notion will be used frequently in our proofs.

Definition 2.1 Let c, ϵ be positive reals where $0 < \epsilon < 1$. A graph H is called (c, ϵ)-*dense* if $e(H) \geq c[n(H)]^{1+\epsilon}$.

In our proofs, we will often use the well-known fact that a graph G satisfying $e(G) \geq d \cdot n(G)$ contains an induced subgraph of minimum degree at least d. The original statement of next lemma is slightly weaker and the proof is probabilistic. The version we present here is suggested by Kostochka [11].

Lemma 2.2 *Let a, m, q be positive integers. Let $A_1, \ldots, A_m$ be a collection of sets of size a. Suppose each elements of $A = \bigcup_i A_i$ lies in at most q different A_i's. Then for each $i \in [m]$, there exists $B_i \subseteq A_i$ of size $\lfloor a/q \rfloor$ such that $B_1, \ldots, B_m$ are pairwise disjoint.*

Proof. Let $p = \lfloor a/q \rfloor$. Create a bipartite graph H with a bipartition (X, A) where $|X| = mp$ as follows. Label the vertices of X by $x_1^1, \ldots, x_1^p, x_2^1, \ldots, x_2^p, \ldots, x_m^1, \ldots, x_m^p$. For each $i \in [m]$ and $y \in A$, if $y \in A_i$ then we add edges between y and $x_i^1, \ldots, x_i^p$. By our construction, each vertex in X has degree a. Also, since each $y \in A$ lies in at most t different A_i's, each vertex in A has degree at most $pq \leq a$ in H. By Hall's Theorem, it is easy to see that H has a matching M saturating all of X. For each $i \in [m]$, the elements of A that $x_i^1, \ldots, x_i^p$ are matched to by M are elements of A_i. Hence, we obtain disjoint $B_1, \ldots, B_m$ each of size $p = \lfloor a/q \rfloor$, with $B_i \subseteq A_i$ for each $i \in [m]$. ■

In the following lemma and theorem, in order to simplify the presentation while not affecting the final main proof, we assume c to be a constant real that is at least 1 and G to be a bipartite graph. Similar statements still hold when c is any positive real and when G is any graph. The next technical lemma provides the most crucial ingredient of our proof of Theorem 1.10.

Lemma 2.3 *Let t be a positive integer and c, ϵ positive reals, where $0 < \epsilon < \frac{1}{2}$ and $c \geq 1$. Let n be a sufficiently large positive integer (depending on t and c, ϵ). Let G be an n-vertex bipartite graph with minimum degree at least cn^ϵ. Let $l(\epsilon) = \lfloor \frac{5}{\epsilon} \ln \frac{1}{\epsilon} \rfloor$. Suppose that G has no $(\frac{c}{2^6}, \epsilon)$-dense subgraph of radius at most $l(\epsilon)$. Then for each vertex u in G, there are at least $n^{1-\frac{\epsilon^2}{2(1+\epsilon)}}$ vertices x each of which is joined to u by at least $l(\epsilon) \cdot t^2$ internally disjoint paths of length at most $l(\epsilon)$.*

Proof. First we show that G has good expansion properties.

Claim 1. Let H be a subgraph of G of radius at most $l(\epsilon) - 1$, where $n(H) \leq 3n^a$ and $a < 1$. Let W be a set of vertices outside H having neighbors in H. Suppose G contains at least $\frac{c}{4} n^{a+\epsilon}$ edges between $V(H)$ and W. Then $|W| \geq 2n^{a + \frac{\epsilon}{1+\epsilon}(1-a)}$.

Proof of Claim 1. Let F denote the subgraph of G induced by $V(H) \cup W$. Since H has radius at most $l(\epsilon) - 1$, F has radius at most $l(\epsilon)$. By our assumption of G, F is not $(\frac{c}{2^6}, \epsilon)$-dense. Suppose $n(F) = n^b$. We have

$$\frac{c}{4} n^{a+\epsilon} \leq e(F) \leq \frac{c}{2^6} (n^b)^{1+\epsilon} \tag{1}$$

From this, we get

$$n^b \geq 16^{\frac{1}{1+\epsilon}} n^{\frac{a+\epsilon}{1+\epsilon}} \geq 5n^{a+\frac{\epsilon}{1+\epsilon}(1-a)}.$$

Hence, $|W| = n(F) - n(H) \geq 5n^{a+\frac{\epsilon}{1+\epsilon}(1-a)} - 3n^a \geq 2n^{a+\frac{\epsilon}{1+\epsilon}(1-a)}$. ■

Let $d = \lceil cn^\epsilon \rceil$. We know that each vertex of G has degree at least d. Let u be any vertex in G. We iteratively define a sequence of disjoint sets $L_1, L_2, \ldots, L_{l(\epsilon)-1}$. During our procedure, we will maintain the following conditions:

(1) Each L_i is called a *level* and will be designated as *strong* or *weak*. Set L_1 will be strong.

(2) Each L_i is partitioned into some d_i subsets L_i^j called *sectors* of equal size. If L_i is a strong level then each sector consists only of a single vertex. If L_i is a weak level, then $\frac{1}{2}d_{i-1} \leq d_i \leq d_{i-1}$.

(3) Each vertex in a strong level L_i beyond L_1 has neighbors in at least $l(\epsilon) \cdot t^2$ sectors of L_{i-1}.

(4) If L_i is a weak level, then there exists an injection f from the collection of sectors of L_i into the collection of sectors of L_{i-1} such that if L_i^j is a sector of L_i then each vertex in L_i^j has at least one neighbor in $f(L_i^j)$. We call $f(L_i^j)$ the *parent sector* of L_i^j in L_{i-1}.

(5) For each i, suppose $|L_i| = n^{a_i}$. If L_{i+1} is a strong level, then $|L_{i+1}| \geq 2|L_i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i)}$. Thus, $a_{i+1} \geq a_i + \frac{\epsilon}{1+\epsilon}(1-a_i)$. If L_{i+1} is a weak level, $|L_{i+1}| \geq 2|L_i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.9\epsilon)}$. Thus, $a_{i+1} \geq a_i + \frac{\epsilon}{1+\epsilon}(1 - a_i + 0.9\epsilon)$.

Recall that G has minimum degree at least $cn^\epsilon \geq n^\epsilon$. To start, let $L_1 = \{x_1, \ldots, x_{\lceil n^\epsilon \rceil}\}$ be a set of $d_1 = \lceil n^\epsilon \rceil$ neighbors of u. We designate L_1 as a strong level. For each $j \in [d_1]$, let $L_1^j = \{x_j\}$. Suppose $L_1, \ldots, L_i$ have been defined so that (1)-(5) hold for L_j, $j \leq i$. Let $U_{i-1} = \{u\} \cup L_1 \cup L_2 \cup \ldots \cup L_{i-1}$. By (5), for each $j \leq i$, $|L_j| \geq 2|L_{j-1}| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_{j-1})} \geq 2|L_{i-1}|$. So, $|U_{i-1}| \leq |L_{i-1}|(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots) \leq 2|L_{i-1}| \leq |L_i|$. It is clear from conditions (3) and (4) that there is a path from each vertex in L_i back to u through U_{i-1}. Thus there is a subgraph H_i of G containing L_i of radius at most i and order at most $|U_{i-1}| + |L_i| \leq 2|L_i|$.

Let d denote the minimum degree of G. We have $d \geq cn^\epsilon$. Let L_i^* denote the set of vertices in L_i having at least $\frac{d}{2}$ neighbors in U_{i-1}. Since L_i is itself an independent set, each vertex in $L_i - L_i^*$ has at least $\frac{d}{2}$ neighbors outside $U_i = U_{i-1} \cup L_i$. Let H^* be the subgraph of G induced by $U_{i-1} \cup L_i^*$. Then H^* has radius at most $l(\epsilon)$. If $|L_i^*| > \frac{1}{4}|U_{i-1}|$, then $|L_i^*| \geq \frac{n(H^*)}{5}$ and $e(H^*) \geq |L_i^*| \cdot \frac{d}{2} \geq |L_i^*| \cdot \frac{c}{2}n^\epsilon \geq \frac{c}{10}[n(H^*)]^{1+\epsilon}$, contradicting G containing no $(\frac{c}{2^6}, \epsilon)$-dense subgraph of radius at most $l(\epsilon)$. Hence $|L_i^*| \leq \frac{1}{4}|U_{i-1}| \leq \frac{1}{4}|L_i|$.

Consider the d_i sectors L_i^j of L_i. If $|L_i^j \cap L_i^*| > \frac{1}{2}|L_i^j| = \frac{1}{2}\frac{|L_i|}{d_i}$, we say L_i^j is *bad*. Otherwise we say it is *good*. If more than $\frac{d_i}{2}$ of the sectors are bad, then $|L_i^*| > \frac{d_i}{2} \cdot \frac{1}{2}\frac{|L_i|}{d_i} = \frac{1}{4}|L_i|$, a contradiction. So at most $\frac{d_i}{2}$ of the sectors are bad and at least $\frac{d_i}{2}$ of the sectors are good.

Now, let $J = \{j : L_i^j \text{ is a good sector of } L_i\}$. For each $j \in J$, let W_i^j denote the set of neighbors of L_i^j outside U_i and suppose $|L_i^j| = n^{a_{i,j}}$.

Claim 2. For each $j \in J$, we have $|W_i^j| \geq 2n^{a_{i,j} + \frac{\epsilon}{1+\epsilon}(1-a_{i,j})} = 2|L_i^j| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}$.

Proof of Claim 2. Let $j \in J$. By our assumption, at least $\frac{1}{2}|L_i^j|$ vertices in L_i^j have at least $\frac{d}{2}$ neighbors outside U_i. If L_i is a strong level, then L_i^j is a single vertex v and v has at least $\frac{d}{2}$ neighbors outside U_i. So $|W_i^j| \geq \frac{d}{2} \geq \frac{c}{2}n^\epsilon \geq 2|L_i^j| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}$, for large n, noting that $|L_i^j| = 1$.

Suppose now that L_i is a weak level. Then L_i^j has a parent sector $L_{i-1}^{j'}$ in L_{i-1} of size $|L_{i-1}|/d_{i-1}$. Since $d_i \le d_{i-1}$, we have $|L_i^j|/|L_{i-1}^{j'}| \ge |L_i|/|L_{i-1}| \ge 2n^{\frac{\epsilon}{1+\epsilon}(1-a_{i-1})} \ge 2$. By backtracking, we can find a tree H_i^j containing L_i^j that is rooted at u and has depth $i \le l(\epsilon)-1$ and order at most $|L_i^j|(1+\frac{1}{2}+\frac{1}{2^2}+\cdots)+l(\epsilon) \le 2|L_i^j|+l(\epsilon)$. (Indeed, a sector in a weak level has a parent sector in the previous level that is much smaller, while any sector in a strong level is linked to u by a path of length at most $l(\epsilon)$.) If $a_{i-1} \le \frac{3\epsilon}{2} < \frac{3}{4}$, then $|L_i^j|/|L_{i-1}^{j'}| \ge 2n^{\frac{\epsilon}{1+\epsilon}\cdot\frac{1}{4}}$, which implies $|L_i^j| \ge l(\epsilon)$ for large n. If $a_{i-1} \ge \frac{3\epsilon}{2}$ instead, then $|L_i^j| \ge 2|L_{i-1}^{j'}| = 2|L_{i-1}|/d_{i-1} \ge 2n^{\frac{3\epsilon}{2}}/\lceil n^\epsilon \rceil \ge l(\epsilon)$, for large n. Hence, we always have $|L_i^j| \ge l(\epsilon)$. So H_i^j has order at most $2|L_i^j|+l(\epsilon) \le 3|L_i^j|$.

By our assumption, at least $\frac{1}{2}|L_i^j|$ vertices in L_i^j have at least $\frac{d}{2}$ neighbors outside U_i. So, there are at least $\frac{1}{2}|L_i^j|\cdot\frac{cn^\epsilon}{2} = \frac{c}{4}n^{a_{i,j}+\epsilon}$ edges between H_i^j and W_i^j. Since $n(H_i^j) \le 3|L_i^j| = 3n^{a_{i,j}}$ and H_i^j has radius at most $l(\epsilon)-1$, applying Claim 1 with $H = H_i^j$ and $W = W_i^j$, we have $|W_i^j| \ge 2n^{a_{i,j}+\frac{\epsilon}{1+\epsilon}(1-a_{i,j})} = 2|L_i^j|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}$. ∎

Let $W_i = \bigcup_{j\in J} W_i^j$. Let $q = t^2\cdot l(\epsilon)$. We say a vertex y in W_i is *heavy* if it belongs to at least q different W_i^j's. Otherwise we say y is *light*. Let W_i^+ denote the set of heavy vertices in W_i and W_i^- the set of light vertices in W_i. We consider two cases. Recall that there is a subgraph H_i containing L_i that has radius at most $i \le l(\epsilon)-1$ and order at most $2|L_i|$.

Case 1. At least half of the edges between L_i and W_i are incident to W_i^+.

In this case, we let $L_{i+1} = W_i^+$ and designate it as a strong level. By our earlier discussion, at most $\frac{1}{4}|L_i|$ vertices in L_i are bad. So, there are at least $\frac{3}{4}|L_i|\cdot\frac{cn^\epsilon}{2} \ge \frac{3}{8}|L_i|\cdot cn^\epsilon = \frac{3c}{8}n^{a_i+\epsilon} \ge \frac{c}{4}n^{a_i+\epsilon}$ edges between L_i and L_{i+1}. Recall that $|L_i| = n^{a_i}$, H_i contains L_i, $n(H_i) \le 2|L_i| = 2n^{a_i}$, and H_i has radius at most $l(\epsilon)-1$. Applying Claim 1 with $H = H_i$, $a = a_i$, $W = |L_{i+1}|$, we have $|L_{i+1}| \ge 2n^{a_i+\frac{\epsilon}{1+\epsilon}(1-a_i)} = 2|L_i|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i)}$. Let $d_{i+1} = |L_{i+1}|$. We partition L_{i+1} into d_{i+1} sectors each consisting of a single vertex. By our assumption, each vertex y in $L_{i+1} = W_i^+$ has neighbors in at least q different sectors in L_i. So (1)-(5) hold for L_{i+1}.

Case 2. At least half of the edges between L_i and W_i are incident to W_i^-.

By our assumption, each vertex y in W_i^- belongs to at most q different W_i^j's. By Claim 2, for each $j \in J$, $|W_i^j| \ge 2|L_i^j|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}$; let Z_i^j denote a subset of W_i^j of size $2|L_i^j|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}$. Note that $|Z_i^j|$ is the same for all $j \in J$, since $|L_i^j|$ is the same for all j. By Lemma 2.2, for each $j \in J$, there exists a subset $A_i^j \subseteq Z_i^j$ of size $|Z_i^j|/q$ such that the A_i^j's so obtained are pairwise disjoint. We let $L_{i+1} = \bigcup_{j\in J} A_i^j$. Let $d_{i+1} = |J|$. The A_i^j's for $j \in J$ form a partition of L_{i+1} into d_{i+1} many subsets of equal size. We define them to be the sectors of L_{i+1} and rename them $L_{i+1}^1, \ldots, L_{i+1}^{d_{i+1}}$, respectively. We will designate L_{i+1} as a weak level. It remains to verify that (2), (4) and (5) hold for L_{i+1}. We have $d_{i+1} = |J| \ge \frac{1}{2}d_i$. So (2) holds. For (4), if $L_{i+1}^j = A_i^{j'} \subseteq W_i^{j'}$, we let $f(L_{i+1}^j) = L_i^{j'}$. It is readily seen that such f satisfies (4).

Since (1), (2) and (5) hold for $L_1, \ldots, L_i$, it is easy to see that $d_i \ge d_1/2^i = n^\epsilon/2^i$. Hence $|L_i|/|L_i^j| = d_i \ge n^\epsilon/2^i \ge n^{0.95\epsilon}$, for large n. This implies $a_i - a_{i,j} \ge 0.95\epsilon$ or $a_{i,j} \le a_i - 0.95\epsilon$. So, $|Z_i^j| \ge 2|L_i^j|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.95\epsilon)} \ge 4q\cdot|L_i^j|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.9\epsilon)}$, for large n. Hence, $|A_i^j| = |Z_i^j|/q \ge 4|L_i^j|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.9\epsilon)}$. Since $|J| \ge \frac{d_i}{2}$, we have $|L_{i+1}| = \sum_{i\in J}|A_i^j| \ge \frac{d_i}{2}\cdot 4|L_i^j|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.9\epsilon)} = 2|L_i|\cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.9\epsilon)}$. So (5) holds for L_{i+1}.

We have now constructed the sequence $L_1, L_2, \ldots,$ that satisfy conditions (1)–(5).

Claim 3. Each vertex x in a strong level L_i is joined to u by at least q internally disjoint paths of length i.

Proof of Claim 3. By (3), x has neighbors in at least q sectors of L_{i-1}. Let $y^1_{i-1}, \ldots, y^q_{i-1}$ be neighbors of x in L_{i-1} all from different sectors. If L_{i-1} is a weak level, then by condition (4), the q sectors involving $y^1_{i-1}, \ldots, y^q_{i-1}$ all have different parent sectors in L_{i-2}. So, for each $j \in [q]$, we can find a neighbor y^j_{i-2} of y^j_{i-1} in L_{i-2} so that $y^1_{i-2}, \ldots, y^q_{i-2}$ are from different sectors of L_{i-2}. If L_{i-1} is a strong level instead and $i - 1 > 1$, then by (3) each y^j_{i-1}, $j \in [q]$, has neighbors in at least q different sectors of L_{i-2}. In this case, it is easy to see that we can still find $y^1_{i-2}, \ldots, y^q_{i-2}$, all from different sectors of L_{i-2}, such that y^j_{i-2} is a neighbor of y^j_{i-1} for each $j \in [q]$. We can continue this process and build q internally disjoint paths of length i from x back to u. ∎

By (5), for each $i \leq l(\epsilon) - 1$ we have $a_{i+1} \geq a_i + \frac{\epsilon}{1+\epsilon}(1 - a_i)$. Solving the recurrence, with $a_1 \geq \epsilon$, we get $a_i \geq 1 - \frac{1-\epsilon}{(1+\epsilon)^{i-1}}$. Let $m = \min\{i : a_i \geq 1 - \frac{\epsilon^2}{2(1+\epsilon)}\}$. To have $a_i \geq 1 - \frac{\epsilon^2}{2(1+\epsilon)}$, it suffices that $1 - \frac{1-\epsilon}{(1+\epsilon)^{i-1}} \geq 1 - \frac{\epsilon^2}{2(1+\epsilon)}$ which holds if $(i-2)\ln(1+\epsilon) \geq \ln\frac{2}{\epsilon^2}$ $(*)$. Since we assume $0 < \epsilon < 0.5$, one can check that $\ln(1+\epsilon) \geq 0.8\epsilon$. In order for $(*)$ to hold, it suffices to have $(i-2)(0.8\epsilon) \geq \ln\frac{1}{\epsilon^3}$, from which we get $i \geq 2 + \frac{1}{0.8\epsilon}\ln\frac{1}{\epsilon^3}$. It follows that $m \leq 2 + \frac{3}{0.8\epsilon}\ln\frac{1}{\epsilon} \leq 2 + \frac{4}{\epsilon}\ln\frac{1}{\epsilon} \leq l(\epsilon) - 2$.

Claim 4. L_{m+1} is a strong level.

Proof of Claim 4. Suppose instead that L_{m+1} is a weak level. By condition (5), we have $|L_{m+1}| \geq 2|L_m| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_m+0.9\epsilon)} \geq 2|L_m| \cdot n^{0.9\frac{\epsilon^2}{1+\epsilon}} \geq 2n^{1-\frac{\epsilon^2}{2(1+\epsilon)}+\frac{0.9\epsilon^2}{1+\epsilon}} \gg n$, a contradiction. So, L_{m+1} must be a strong level. ∎

Now, by our choice of m and (5), $|L_{m+1}| \geq 2n^{1-\frac{\epsilon^2}{2(1+\epsilon)}} \geq n^{1-\frac{\epsilon^2}{2(1+\epsilon)}}$. By Claim 3, each vertex x in L_{m+1} is joined to u by at least $q = t^2 l(\epsilon)$ internally disjoint paths of length $m + 1 \leq l(\epsilon)$. Since u is arbitrary, this proves the lemma. ∎

Theorem 2.4 *Let t be a positive integer and c, ϵ fixed reals, where $0 < \epsilon < \frac{1}{2}$ and $c \geq 1$. Let n be a sufficiently large positive integer (depending on t and c, ϵ). Let G be an n-vertex bipartite graph with minimum degree at least cn^ϵ. Let $l(\epsilon) = \lfloor \frac{5}{\epsilon}\ln\frac{1}{\epsilon} \rfloor$. Suppose G has no $(\frac{c}{2^6}, \epsilon)$-dense subgraph of radius at most $l(\epsilon)$. Then G contains a $2l(\epsilon)$-subdivision of K_t.*

Proof. By Lemma 2.3, for each vertex x in G, there are at least $n^{1-\frac{\epsilon^2}{2(1+\epsilon)}}$ vertices y each of which is joined to x by at least $l(\epsilon) \cdot t^2$ internally disjoint paths of length at most $l(\epsilon)$.

Let's define a new graph H with $V(H) = V(G)$ such that $xy \in E(H)$ if and only if x and y are joined by at least $l(\epsilon) \cdot t^2$ internally disjoint paths of length at most $l(\epsilon)$ in G. By our earlier discussion, each vertex in H has degree at least $n^{1-\frac{\epsilon^2}{2(1+\epsilon)}} \gg t^2 \cdot n^{1/2}$, for large n. Thus, $e(H) \geq \frac{t^2}{2}n^{3/2}$. By Proposition 1.9, H contains a 2-subdivision F of K_t. From F we obtain an $l(\epsilon)$-subdivision M of K_t in G as follows. Let $p = n(F) = t + \binom{t}{2} < t^2$. Suppose $V(F) = \{x_1, \ldots, x_p\}$ and $E(F) = \{e_1, e_2, \ldots, e_q\}$. Initially, let $V(M) = V(F)$ and $E(M) = \emptyset$. Suppose without loss of generality that e_1 joins x_1 and x_2 in $F \subseteq H$. By our definition of H, there exist at least $t^2 \cdot l(\epsilon)$ internally disjoint paths in G between x_1 and x_2.

Since $t^2 \cdot l(\epsilon) > n(F)$, one of these paths P_1 avoids $V(F) - \{x_1, x_2\}$. In other words, P_1 is a path of length at most $l(e)$ in G between x_1 and x_2 that intersects $V(F)$ only at x_1 and x_2. Now let $V(M) = V(M) \cup V(P_1)$ and $E(M) = E(M) \cup E(P_1)$.

In general, suppose we have processed $e_1, e_2, \ldots, e_i$ and for each $j = 1, \ldots, i$, we have added to M some path P_j in G of length at most $l(\epsilon)$ that joins the two endpoints of e_j and intersects the previous M only at these two endpoints. Now, suppose e_{i+1} joins x_a and x_b in F. By definition, there exist at least $t^2 \cdot l(\epsilon)$ internally disjoint paths in G between x_a and x_b. Note that at this point $|V(M)| \leq t + \binom{t}{2} \cdot l(\epsilon) < t^2 \cdot l(\epsilon)$. Thus we can find one of these paths P_{i+1} that avoids $V(M) - \{x_a, x_b\}$. That is, P_{i+1} intersects M only at x_a and x_b. Let $V(M) = V(M) \cup V(P_{i+1})$ and $E(M) = E(M) \cup E(P_{i+1})$. We continue like this till all the edges of F is processed. It is easy to see that the final M is an $l(e)$-subdivision of F in G, which also forms a $2l(\epsilon)$-subdivision of K_t in G.

■

Remark 2.5 From the proof of Theorem 2.4, one can see that a weaker version of Lemma 2.3 will suffice. Indeed, we only need to ensure that for each vertex u there are say at least $t^2 n^{1/2}$ vertices x each of which is joined to u by at least $t^2 l(\epsilon)$ internally disjoint paths. However, in the proof of Lemma 2.3 we deliberately argued that there are least $n^{1-\frac{\epsilon^2}{2(1+\epsilon)}}$ such x as a way to ensure that there are at least $t^2 n^{1/2}$ such x. In fact, this was what introduced the $\ln \frac{1}{\epsilon}$ factor in Theorem 1.10. So, in order to improve Theorem 1.10, one will likely have to improve the arguments used in the proof of Lemma 2.3.

3 Proof of Theorem 1.10

In this section, we prove Theorem 1.10. Let us first recall some facts from [12]. For a vertex u in a graph G, and a positive integer i, let $D_i^u(G) = \{x \in V(G) : d_G(u, x) = i\}$. In other words, $D_i^u(G)$ is the set of vertices at distance i from u in G. When the host graph G is clear, we write D_i^u for $D_i^u(G)$. The following lemma was proved in [12]. For completeness, we include its short proof.

Lemma 3.1 **[12]** *Let c, ϵ be positive reals, where $0 < \epsilon < 1$. Let G be a (c, ϵ)-dense graph and u a vertex in G. Then there exists an i such that $G[D_i^u \cup D_{i+1}^u]$ is $(\frac{c}{2}, \epsilon)$-dense.*

Proof. Suppose there is no such i. For each i let $d_i = |D_i^u \cup D_{i+1}^u|$. We have $e(G[D_i^u \cup D_{i+1}^u]) < (c/2)(d_i)^{1+\epsilon} < (c/2) d_i n^\epsilon$ for all i, where $n = n(G)$. This yields

$$e(G) \leq \sum_i e(G[D_i^u \cup D_{i+1}^u]) < cn^\epsilon (1/2) \sum_i d_i \leq cn^{1+\epsilon},$$

contradicting G being (c, ϵ)-dense. ■

The following simple observation somewhat surprisingly played a key role in [12].

Observation 3.2 Let H be a bipartite graph with a bipartition (X, Y). Suppose there is a path P of $2m$ vertices in H, where the vertices on P are $a_1, b_1, a_2, b_2, \ldots, a_m, b_m$ in order. Then either all the a_i's or all the b_i's are contained in X.

Now, we are ready to prove our main result.

Proof of Theorem 1.10: Let $l(\epsilon) = \frac{5}{\epsilon} \ln \frac{1}{\epsilon}$. It is well-known that G contains a spanning bipartite subgraph G' with $e(G') \geq \frac{1}{2}e(G) \geq 2^{7t^2-1} \cdot n^{1+\epsilon}$. Let $G_0 = G'$, we iteratively define a sequence of subgraphs of G' as follows. Note that these graphs are all bipartite since G' is bipartite. For convenience let $\beta = 2^{7t^2-1}$. Let G_1 be a $(\beta/2^6, \epsilon)$-dense subgraph of G_0 of radius at most $l(\epsilon)$ with center u_1, if it exists. By Lemma 3.1, for some two consecutive distance classes X_1, Y_1 from u_1 in G_1 the subgraph H_1 induced by them is $(\beta/2^7, \epsilon)$-dense, where X_1 denotes the distance class of the two that is closer to u_1. Note that since G_1 is bipartite, each distance class from u_1 is an independent set. Thus, (X_1, Y_1) is a bipartition of H_1. If it exists, let G_2 be a subgraph of H_1 that is $(\beta/2^{13}, \epsilon)$-dense and has radius at most $l(\epsilon)$. Let u_2 be a vertex in the center of G_2. By Lemma 3.1, for some two consecutive distance classes X_2, Y_2 from u_2 in G_2 the subgraph H_2 induced by them is $(\beta/2^{14}, \epsilon)$-dense, where X_2 denotes the distance class of the two that is closer to u_2. As before, (X_2, Y_2) forms a bipartition of H_2. We continue like this. Suppose we have defined $G_1, H_1, G_2, H_2, \ldots, G_i, H_i$, where G_i is $(\beta/2^{7i-1}, \epsilon)$-dense and has radius at most $l(\epsilon)$ and u_i is a vertex in the center, and H_i is a subgraph of G_i induced by some two consecutive distance classes from u_i and is $(\beta/2^{7i}, \epsilon)$-dense. If it exists, let G_{i+1} be a subgraph of H_i that is $(\beta/2^{7i+6}, \epsilon)$-dense and has radius at most $l(\epsilon)$; let u_{i+1} be a vertex in its center. Then for some two consecutive distance classes X_{i+1}, Y_{i+1} of G_{i+1} from u_{i+1} the subgraph H_{i+1} induced by them is $(\beta/2^{7(i+1)}, \epsilon)$-dense, where X_{i+1} is closer to u_{i+1} than Y_{i+1} and (X_{i+1}, Y_{i+1}) forms a bipartition of H_{i+1}.

Clearly, this process eventually terminates, suppose G_m, H_m are the last graphs in the sequence. We consider two main cases.

Case 1. $m \geq t(t-1)$.

Let $s = t(t-1)$. Since H_s is $(\beta/2^{7s}, \epsilon)$-dense, in particular $e(H_s)/n(H_s) \geq \beta n^\epsilon/2^{7s} \geq 2^{t-1}n^\epsilon \geq 2t$. Thus, H_s contains a subgraph F with $\delta(F) \geq 2t$. F contains a path P on $2t$ vertices. Since $H_1 \supseteq H_2 \supseteq \cdots \supseteq H_s$, P lies in all of $H_1, \ldots, H_s$. By our construction for each i, H_i is a bipartite subgraph of G_i induced by some two consecutive distance classes X_i, Y_i from the center u_i where (X_i, Y_i) forms a bipartition of H_i and X_i is the distance class of the two that is closer to u_i.

Let $a_1, b_1, \ldots, a_t, b_t$ be the vertices on P in order. By Observation 3.2, for each $i \in [s]$ we have either $a_1, \ldots, a_t \in X_i$ or $b_1, \ldots, b_t \in X_i$. Let I denote the set of indices i for which the first scenario occurs and I' the set of indices for which the second scenario occurs. Without loss of generality, we may assume that $|I| \geq s/2$.

For each $i \in I$, we can connect any two $a_x, a_y \in \{a_1, \ldots, a_t\}$ by a path $Q_{x,y}$ of length at most $2\,\mathrm{rad}(G_i) \leq 2l(\epsilon)$ through the center u_i such that $V(Q_{x,y}) \cap V(G_{i+1}) = \{a_x, a_y\}$. Using G_i, $i \in I$. we can find such $Q_{x,y}$ for all $x, y \in \{1, \ldots, t\}, x \neq y$. These paths have no common internal vertices and their union is a subdivision of K_t in which each edge of K_t is replaced by a path of length at most $2l(\epsilon) = \frac{10}{\epsilon} \ln \frac{1}{\epsilon}$.

Case 2. $m < t(t-1)$.

By our assumption, H_m is $(\beta/2^{7m}, \epsilon)$-dense but contains no $(\beta/2^{7m+6}, \epsilon)$-dense subgraph of radius at most $l(\epsilon)$, since otherwise G_{m+1} would have been defined. Since $e(H_m) \geq \frac{\beta}{2^{7m}}[n(H_m)]^\epsilon \cdot n(H_m)$, it contains a subgraph F with minimum degree at least $\frac{\beta}{2^{7m}}[n(H_m)]^\epsilon \geq \frac{\beta}{2^{7m}}[n(F)]^\epsilon$. Also, since $F \subseteq H_m$, F contains no $(\beta/2^{7m+6}, \epsilon)$-dense subgraph of radius at most

$l(\epsilon)$. By Corollary 2.4, with $c = \frac{\beta}{2^{7m}} \geq \frac{2^{7t^2-1}}{2^{7t(t-1)}} \geq 1$, F contains a $2l(\epsilon)$-subdivision of K_t. Hence, G contains a $\frac{10}{\epsilon} \ln \frac{1}{\epsilon}$-subdivision of K_t. ■

4 Concluding Remarks

It would be interesting to see if the $\ln \frac{1}{\epsilon}$ factor in Theorem 1.10 can be eliminated. As mentioned in Remark 2.5, this likely requires an improvement of Lemma 2.3. It would be interesting to study $ex(n, H^{(\leq p)})$ for general graphs H. One could get general bounds using our approach for $ex(n, K_t^{(\leq p)})$, but it is conceivable that for special classes of graphs such as graphs with bounded maximum degree sharper bounds can be obtained.

One could similarly define $H^{(p)}$ to be the graph obtained from H by subdividing each edge of H exactly $p-1$ times and study $ex(n, H^{(p)})$. One earlier result of this type was obtained by Faudree and Simonovits [8], described as follows. Let $C_{m,k}$ denote the graph obtained from joining two vertices x and y by m internally disjoint paths of length k. Faudree and Simonovits [8] showed for fixed m and k that $ex(n, C_{m,k}) = O(n^{1+1/k})$, extending the classic result of Bondy and Simonovits [4] that $ex(n, C_{2k}) = O(n^{1+1/k})$. We may view $C_{m,k}$ as obtained from the graph consisting of two vertices joined by m multiple edges by subdividing each of those edges $k-1$ times.

In general, there are very few results on graphs whose Turán number is $O(n^{1+\epsilon})$ for ϵ close to 0. It is easy to see by the usual Turán lower bound in terms of density that such a graph H must have average degree close to 2. Since degree 1 vertices also don't affect the Turán number substantially such a graph must then have most of its vertices having degree 2. This means H is some sort of "long" subdivision of another graph. This observation somewhat indicates that studying p-subdivisions is a meaningful step towards the study of graphs whose Turán number is $O(n^{1+\epsilon})$ for small ϵ.

References

[1] N. Alon, M. Krivelevich, B. Sudakov: Turán numbers of bipartite graphs and related Ramsey-type questions, *Combinatorics, Probability, and Computing*, **12**, 477–494.

[2] B. Bollobás: **Extremal graph theory**, Academic Press, London-New York, 1978.

[3] B. Bollobás, A. Thomason: Proof of a conjecture of Mader, Erdős and Hajnal on topological subgraphs, *European J. Combinatorics* **19** (1998), 883-887.

[4] A. Bondy, M. Simonovits: Cycles of even length in graphs, *J. Combinatorial Theory Ser. B* **16** (1974), 97-105.

[5] P. Erdős: Some unsolved problems in graph theory and combinatorial analysis, *Combinatorial Mathematics and its Applications (Proc. Conf. Oxford* (1969), 97-109. Academic Press London, 1971.

[6] P. Erdős, R. Faudree, C. Rousseau, R. Schelp: On cycle-complete graph Ramsey numbers, *J. Graph Theory*, **2** (1978), 53–64.

[7] P. Erdős, M. Simonovits: Some extremal problems in graph theory, *Combinatorial Theory and Its Applications* **1** (Proc. Colloq. Balatonfüred, 1969), North Holland, Amsterdam, (1970), 377-390.

[8] R. Faudree, M. Simonovits: On a class of degenerate extremal graph problems, *Combinatorica*, **3**(1983), 83–93.

[9] Z. Füredi: On a Turán type problem of Erdős,*Combinatorica* **11** (1991), 75-79.

[10] J. Komós, E. Szemerédi: Topological cliques in graphs. II., *Combinatorics, Probability and Computing* **5** (1996), 79-90.

[11] A. Kostochka: personal communications.

[12] A. Kostochka, L. Pyber: Small topological complete subgraphs of "dense" graphs, *Combinatorica*, **8** (1988), 83–86.

[13] W. Mader: Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Ann.* **174** (1967) 265-268.

[14] B. Sudakov: A note on odd cycle-complete graph Ramsey numbers, *Electronic Journal of Combinatorics*, **9** (2002), N1, 4pp.

[15] A. Thomason: The extremal function for complete minors, *J. Combinatorial Theory Ser B* **81** (2001), 318-338.

[16] D. B. West: **Introduction to graph theory**, second edition, Prentice Hall, 2001.