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On a Conjecture about Trees in Graphs with Large Girth

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The *girth* of a graph G is the length of a shortest cycle in G. Dobson (1994, Ph.D. dissertation, Louisiana State University, Baton Rouge, LA) conjectured that every graph G with girth at least $2t+1$ and minimum degree at least k/t contains every tree T with k edges whose maximum degree does not exceed the minimum degree of G. The conjecture has been proved for $t \leqslant 3$. In this paper, we prove Dobson's conjecture. © 2001 Elsevier Science

Key Words: trees; girth.

1. INTRODUCTION

In 1964, Erdős and Sós made the following well-known conjecture.

Conjecture 1. Every graph of order n with more than $n(k-1)/2$ edges contains every tree with k edges as a subgraph.

The conjecture has been verified only for a few special cases. In 1996, Brandt and Dobson [1] proved the conjecture for graphs with girth at least 5.

THEOREM A. *Every graph of order n and girth at least* 5 *with more than* $n(k-1)/2$ *edges contains every tree with k edges as a subgraph.*

This follows from their slightly stronger result

THEOREM B. *Let G be a graph with girth at least* 5 *and T be a tree with k edges. If* $\delta(G) \geqslant k/2$ *and* $\Delta(G) \geqslant \Delta(T)$, *then G contains T as a subgraph.*

By replacing $\Delta(G) \geqslant \Delta(T)$ with the stronger condition $\delta(G) \geqslant \Delta(T)$, Dobson [2] made the following general conjecture.

Conjecture 2. Let G be a graph with girth at least $2t+1$ and T be a tree with k edges. If $\delta(G) \geqslant k/t$ and $\delta(G) \geqslant \Delta(T)$, then G contains T as a subgraph.

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Note that if the condition $\delta(G) \geqslant \Delta(T)$ is replaced with $\Delta(G) \geqslant \Delta(T)$, then the conjecture does not hold for $t \geqslant 3$. For instance, for $t=3$, $k \geqslant 9$, let T be the double star with k edges in which the two non-leaf vertices have degree $\lfloor \frac{k+1}{2} \rfloor$ and $\lceil \frac{k+1}{2} \rceil$, respectively, and define G as follows. Let H be a $\lceil k/3 \rceil$-regular graph with girth 7. Form G from a disjoint union of at least $\lceil \frac{k+1}{2} \rceil$ copies of H by adding a vertex, adjacent to exactly one vertex in each copy of H. It is easy to check that G has girth 7, maximum degree at least $\lceil \frac{k+1}{2} \rceil \geqslant \Delta(T)$ and minimum degree at least $\lceil k/3 \rceil$. However, T cannot be embedded in G, since G has only one vertex whose degree is at least $\lfloor \frac{k+1}{2} \rfloor$. This example can clearly be generalized for all $t \geqslant 3$.

Conjecture 2 is known to be true for $t \leqslant 3$. The fact that it holds for $t=1$ is well known (see [7]). The case $t=2$ is implied by Theorem B. The case $t=3$ was proved by Saclé and Woźniak [5].

In this paper, we prove the conjecture. We state the result in the following equivalent form.

THEOREM 1. *Let G be a graph with girth at least $2t+1$ and T be a tree with at most k edges. If $\delta(G) \geqslant k/t$ and $\delta(G) \geqslant \Delta(T)$, then G contains T as a subgraph.*

2. PRELIMINARIES

We first introduce some terminology. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [7].

Given graphs G and H, an *embedding* of H in G is an injection f: $V(H) \to V(G)$ such that $uv \in E(H)$ implies $f(u)\, f(v) \in E(G)$. Let T be a tree; a *leaf* in T is a vertex with degree 1. Two leaves with the same neighbor in T are *siblings*. The *derived tree* of T, denoted by $D(T)$, is the subtree of T obtained by deleting all the leaves of T. For a subtree T' of $D(T)$, $L(T')$ denotes the subtree of T containing T' and all the leaves in T that are adjacent to $V(T')$. A *penultimate vertex* in T is a leaf in $D(T)$. For a positive integer m, $[m]$ denotes the set of integers 1, ..., m. A component in a graph is *nontrivial* if it contains at least two vertices. If P is a path and x, y are two vertices on P, then $P[x, y]$ denotes the portion of P between x and y.

Next, we develop some useful tools in the following lemmas. In our proofs of the lemmas and Theorem 1, we will implicitly use the fact that a closed walk containing an edge that is used only once necessarily contains a cycle. For the rest of the paper, we assume that $t \geqslant 2$. We begin with an easy observation. We omit the proof since it is straightforward.

LEMMA 1. *Let G be a graph with girth at least $2t+1$ and T be a tree with diameter at most $2t$. If $\delta(G) \geqslant \Delta(T)$, then G contains T as a subgraph.*

Furthermore, if x, y are two adjacent vertices in G and u, v are two adjacent vertices in T, then T can be embedded in G in such a way that u, v are mapped to x, y respectively.

LEMMA 2. *Let G be a connected graph of order n and S be a subset of $V(G)$ such that every pair in S has distance at least $2t-1$ in G. Then $|S| \leqslant \max\{\lfloor \frac{n}{t} \rfloor, 1\}$.*

Proof. We may assume that S contains at least two vertices. Suppose $S=\{v_1, ..., v_m\}$, where $m \geqslant 2$. For each $i \in [m]$, let $N_i=\{u \in V(G)\colon dist_G(u, v_i) \leqslant t-1\}$. Since G is connected and a shortest path connecting any pair v_i, v_j has length at least $2t-1$, we have $|N_i| \geqslant 1+(t-1)=t$, for each $i \in [m]$, and $N_i \cap N_j = \varnothing$ for $i \neq j$. Hence $n \geqslant |\bigcup_{i=1}^m N_i| = \sum_{i=1}^m |N_i| \geqslant mt$, and therefore $m \leqslant \lfloor \frac{n}{t} \rfloor$. ▮

Lemma 2 is best possible for trees. The tree T obtained by identifying each vertex on a P_m with an endpoint of a P_t has mt vertices and its leaves form a subset of size m with pairwise distance at least $2t-1$ in T.

Fixing k, t, and a graph G with a girth of at least $2t+1$ and a minimum degree of at least k/t, a *minimal nonembeddable tree* is a tree T with at most k edges and maximum degree at most $\delta(G)$, such that T cannot be embedded in G but every proper subtree of T can be embedded in G. The next two lemmas reveal some properties that such a minimal nonembeddable tree would have.

LEMMA 3. *Let T be a minimal nonembeddable tree for some fixed k, t, G. Then T contains at most $t-1$ penultimate vertices.*

Proof. Suppose otherwise that T contains at least t penultimate vertices. Let w be a penultimate vertex of T with the least number of adjacent leaves. Let $v_1, ..., v_l$ be its adjacent leaves. Let $w_1, ..., w_{t-1}$ denote $t-1$ other penultimate vertices in T, each adjacent to at least l leaves. By our assumption, there exists an embedding f of $T' = T - \{v_1, ..., v_l\}$ in G. Let $T'' = T' - w$. Then T'' is a subtree of T with at most $(k+1)-(l+1)=k-l$ vertices. Let S denote the set of neighbors (in G) of $f(w)$ in $V(f(T''))$. Since G has girth at least $2t+1$, the distance in $f(T'')$ between any pair of vertices of S is at least $2t-1$, which is at least 3 for $t \geqslant 2$. In particular, no two sibling leaves in $f(T'')$ can belong to S at the same time. Hence at most one of the leaves adjacent to $f(w_i)$ can be a member of S, for $i \in [t-1]$. This means that for each $i \in [t-1]$ we can delete $l-1$ leaves adjacent to $f(w_i)$ in $f(T'')$ without deleting a member of S. Denote the subtree of $f(T'')$ obtained in this way by T^*. T^* has $n(f(T''))-(t-1)(l-1) \leqslant (k-l)-(t-1)(l-1) = k-tl+(t-1)$ vertices. The distance between every pair in S remains the same in T^* as in $f(T'')$. We may further assume that $n(T^*) \geqslant t$, since otherwise T^* has diameter at most $t-2$ and T would have diameter at most $(t-2)+2=t$, contradicting Lemma 1. By Lemma 2, we have $|S| \leqslant \max\{\lfloor \frac{n(T^*)}{t} \rfloor, 1\} = \lfloor \frac{n(T^*)}{t} \rfloor \leqslant \lfloor \frac{k-tl+(t-1)}{t} \rfloor = \lceil \frac{k}{t} \rceil - l$. Hence $f(w)$ has at most $\lceil \frac{k}{t} \rceil - l$ neighbors in $V(f(T''))$. Since $\delta(G) \geqslant \lceil \frac{k}{t} \rceil$, $f(w)$ has at least l neighbors outside $V(f(T''))$. In that case, we can extend f to an embedding of T in G by embedding $v_1, ..., v_l$ in the neighborhood of $f(w)$ outside $V(f(T''))$, contradicting our assumption. ∎

Lemma 4. *Let T be a minimal nonembeddable tree for some fixed k, t, G. Let x be a vertex in T such that every nontrivial component in $T-x$ has at least t vertices. Then x is adjacent to no leaves of T or to at least two leaves of T. In particular, each penultimate vertex in T is adjacent to at least two leaves.*

Proof. To prove the first part of the claim, suppose that x is adjacent to exactly one leaf v in T. Let $T' = T - v$. By our assumption, there exists an embedding f of T' in G. Let $T_1, ..., T_m$ denote the components in $T'-x$, with $n_1, ..., n_m$ vertices respectively. Each T_i is nontrivial, hence $n_i \geqslant t$. Let S_i denote the set of neighbors of $f(x)$ in $V(f(T_i))$. Since G has girth at least $2t+1$, every pair in S_i has a distance of at least $2t-1$ in $f(T_i)$. By Lemma 2, $|S_i| \leqslant \max\{\frac{n(f(T_i))}{t}, 1\} = \lfloor \frac{n_i}{t} \rfloor \leqslant \frac{n_i}{t}$. Hence $f(x)$ has at most $\lfloor \sum_{i=1}^{m} (n_i/t) \rfloor = \lfloor n(f(T'-x))/t \rfloor \leqslant \lfloor \frac{k-1}{t} \rfloor = \lceil \frac{k}{t} \rceil - 1$ neighbors in $V(f(T'-x))$. Since $\delta(G) \geqslant \lceil \frac{k}{t} \rceil$, $f(x)$ has at least one neighbor outside $V(f(T'-x))$, in which case we can extend f to an embedding of T in G by mapping v to a neighbor of $f(x)$ outside $V(f(T'-x))$, contradicting our assumption. This proves the first part of the claim.

Now, let x be a penultimate vertex of T. By Lemma 1, we may assume that T has diameter at least $2t+1$, in which case $T-x$ consists of isolated vertices and a component of diameter at least $2t-1$; such a component has at least t vertices. By the first part of the claim, x is adjacent to either none or to at least two leaves in T. Since x is adjacent to at least one leaf, x must be adjacent to at least two leaves. ∎

We omit the proof of the next lemma since it is straightforward.

Lemma 5. *A tree with l leaves has at most $l-2$ vertices of degree at least* 3.

3. PROOF OF THEOREM 1

Proof of Theorem 1. Fixing k, t, and a graph G with girth at least $2t+1$ and minimum degree at least k/t, we prove that a minimal nonembeddable tree does not exist. This will prove the theorem. Suppose otherwise that there exists a minimal nonembeddable tree T with at most k edges and maximum degree at most $\delta(G)$. Without loss of generality, we may assume that T has k edges. By Lemma 3 and Lemma 4, T contains at most $t-1$ penultimate vertices, each adjacent to at least two leaves. Hence the derived tree $D(T)$ contains at most $t-1$ leaves. If $D(T)$ does not contain any vertex of degree 2, then by Lemma 5 $D(T)$ contains at most $t-3$ nonleaf vertices, in which case $D(T)$ has diameter at most $t-2$ and T has diameter at most $t \leqslant 2t$, contradicting Lemma 1. Hence we may assume that $D(T)$ contains at least one vertex of degree 2.

Given a vertex x in $D(T)$ and a component A in $D(T)-x$, we define the *depth* of A from x, denoted by $dep(x, A)$, as $\max\{dist_{D(T)}(x, u)\colon u \in V(A)\}$ and define the depth of x, denoted by $dep(x)$, as $\min\{dep(x, A)\colon A$ is a component in $D(T)-x\}$.

Given positive integers a, b with $a \geqslant 2b$, let $R(a, b)$ denote the tree with $2b$ leaves obtained from the path $v_1 \cdots v_a$ by adding a leaf adjacent to each of $v_2, ..., v_b, v_{a-(b-1)}, ..., v_{a-1}$.

CLAIM 1. *Let* $m = \min\{dep(x)\colon x$ *is a vertex of degree* 2 *in* $D(T)\}$. *Then* $m \leqslant (t-1)/2$. *Furthermore, if* $m = (t-1)/2$, *then* $D(T) = R(r, m)$ *for some* $r \geqslant 2m$, *and each vertex of degree* 2 *in* $D(T)$ *is adjacent to either none or at least two leaves in* T.

Proof. Let x be any vertex of degree 2 in $D(T)$ and A_1, A_2 be the two components in $D(T)-x$. Let u be a vertex in A_1 at maximum distance from x in $D(T)$ and v be a vertex in A_2 at maximum distance from x in $D(T)$. Let $P_1 = u_0 u_1 \cdots u_l$ and $P_2 = v_0 v_1 \cdots v_h$ denote the u, x-path and the v, x-path in $D(T)$, respectively, where $u_0 = u$, $v_0 = v$, and $u_l = x = v_h$.

Let $p \leqslant l$ denote the smallest subscript such that u_p has degree 2 in $D(T)$ and q denote the smallest subscript such that v_q has degree 2 in $D(T)$. Let A_p denote the component in $D(T)-u_p$ containing u and A_q denote the component in $D(T)-v_q$ containing v. It is clear, by our choice of u, v, that A_p has depth p from u_p and A_q has depth q from v_q. Hence, $p \geqslant m$, $q \geqslant m$. For $i \in [p-1]$, u_i has degree at least 3 in $D(T)$ and hence leads to at least one leaf in $D(T)$; such a leaf clearly lies in A_p. Since $u_0 = u$ is also a leaf in $D(T)$, A_p contains at least $p \geqslant m$ leaves of $D(T)$ and at least $(p-1)+p = 2p-1 \geqslant 2m-1$ vertices. If A_p contains exactly m leaves, then $p = m$ and A_p cannot contain any vertex having degree 2 in $D(T)$, since such a vertex would have depth less than $p = m$. In that case, it is clear that A_p contains exactly the path $u_0 u_1 \cdots u_{m-1}$ plus one leaf adjacent to each of $u_1, ..., u_{m-1}$.

Similarly, A_q contains at least $q \geqslant m$ leaves of $D(T)$ and at least $2q-1 \geqslant 2m-1$ vertices. If A_q contains exactly m leaves, then $q = m$ and A_q contains exactly the path $v_0 v_1 \cdots v_{m-1}$, plus a leaf adjacent to each of $v_1, ..., v_{m-1}$.

Hence, all together $D(T)$ contains at least $p+q \geqslant 2m$ leaves. Since $D(T)$ contains at most $t-1$ leaves, we have $2m \leqslant t-1$ or $m \leqslant (t-1)/2$. If $m=(t-1)/2$, then $D(T)$ contains exactly $2m$ leaves, which happens only if A_p, A_q each contains exactly m leaves of $D(T)$ and all the vertices in $D(T)-A_p-A_q$ have degree 2 in $D(T)$. In that case, we have $D(T)=R(l+h+1, m)$.

The two nontrivial components in $T-x$ are exactly $L(A_1)$ and $L(A_2)$. $L(A_1)$ contains at least $2m-1$ vertices in A_p and at least two leaves in T adjacent to u, hence $L(A_1)$ contains at least $2m+1$ vertices. Similarly, $L(A_2)$ contains at least $2m+1$ vertices. If $m=(t-1)/2$, then $2m+1=t$. By Lemma 4, x is adjacent to either no leaves of T or at least two leaves of T. ▌

Remark 1. Let x be an arbitrary vertex of degree 2 in $D(T)$. By the definition of m, we have $dep(x) \geqslant m$. Hence, if A is a component in $D(T)-x$ then A has depth at least m from x by the definition of $dep(x)$. (See the corresponding definitions regarding depth in the second paragraph prior to Claim 1.)

Now, let w be a vertex of degree 2 in $D(T)$ with depth m, where m is defined as in Claim 1, and let A_1, A_2 denote the two components of $D(T)-w$, where A_1 has depth m from w. Let w' denote the unique neighbor of w in A_1, and let w'' denote the unique neighbor of w in A_2. We have $dist_{D(T)}(w', x) \leqslant m-1$, $\forall x \in V(A_1)$.

The two components of $D(T)-ww'$ are A_1 and $A_2 \cup ww''$ (adding an edge always includes adding its endpoints). So, the two components of $T-ww'$ are $L(A_1)$ and $L(A_2 \cup ww'')$. Let $T_1 = L(A_1) \cup ww'$, and let $T_2 = L(A_2 \cup ww'') \cup ww'$; T_1 and T_2 are proper subtrees of T with $T_1 \cup T_2 = T$ and $T_1 \cap T_2 = w'w$. Furthermore, since $dist_{D(T)}(w', x) \leqslant m-1$, $\forall x \in V(A_1)$, we have $dist_{T_1}(w', y) \leqslant m$, $\forall y \in V(T_1)$. In particular, this implies that T_1 has diameter at most $2m$.

Each T_i is a proper subtree of T, hence can be embedded in G. We show that we can embed T_1 and T_2 in G in an appropriate way, such that the two embeddings together form an embedding of T in G, which will contradict our assumption about T and complete the proof.

Let f be an embedding of T_2 in G. We define several subsets of $N_G(f(w))$ which count different types of neighbors of $f(w)$. Let $T'' = L(A_2)$ (hence T'' is obtained from T_2 by deleting w and its adjacent leaves in T_2), and let $H = f(T'')$. Let

$$\begin{aligned} N_1 &= N_G(f(w)) \cap V(H), \\ N_2 &= \{u \in N_G(f(w)) - V(H) : dist_{G-f(w)}(u, V(H)) \leqslant m\}, \end{aligned} \tag{1}$$

and

$$N_3 = \{u \in N_G(f(w)) - V(H) : dist_{G-f(w)}(u, V(H)) > m\}.$$

Clearly, $N_G(f(w)) = N_1 \cup N_2 \cup N_3$ and N_1, N_2, N_3 are pairwise disjoint.

Claim 2. *If $N_3 \neq \emptyset$, then T can be embedded in G.*

Proof. Suppose N_3 is nonempty. Note that f maps the leaves adjacent to w in T_2 (including w') into $N_2 \cup N_3$. Rearranging the images of those leaves within $N_2 \cup N_3$ does not change the rest of the embedding or the sets N_1, N_2, N_3. Hence we may assume without loss of generality that $f(w') \in N_3$. (In other words, we may modify f if necessary so that w' is mapped to a vertex in N_3.)

Since T_1 has diameter at most $2m \leqslant 2t$, by Lemma 1 there exists an embedding ϕ of T_1 in G such that $\phi(w') = f(w')$ and $\phi(w) = f(w)$. We show that $V(\phi(T_1)) \cap V(f(T_2)) = \{f(w'), f(w)\}$, in which case $\phi(T_1) \cup f(T_2)$ forms a copy of T in G.

Suppose there exists a vertex $z \in V(\phi(T_1)) \cap V(f(T_2)) - \{f(w'), f(w)\}$. Let $x_1 = \phi^{-1}(z) \in V(T_1) - \{w', w\}$. By our earlier discussion, we have $dist_{T_1}(w', x_1) \leqslant m$. Note that the unique w', x_1-path in T_1 does not use w. Thus the image (under ϕ) of that path is a $\phi(w')$, z-path of length at most m that avoids $\phi(w)$. Hence, we have $dist_{G-f(w)}(\phi(w'), z) \leqslant m$ or, equivalently, $dist_{G-f(w)}(f(w'), z) \leqslant m$.

Let $x_2 = f^{-1}(z) \in V(T_2) - \{w', w\}$. If x_2 is a leaf in T_2 adjacent to w, then the cycle formed by the edges $zf(w)$, $f(w)\,f(w')$ and a shortest $f(w')$, z-path in $G - f(w)$ has length at most $m + 2 \leqslant 2t$, contradicting the girth requirement. Hence $x_2 \in V(T'')$, and therefore $z \in V(H)$. Now, we have $dist_{G-f(w)}(f(w'), V(H)) \leqslant dist_{G-f(w)}(f(w'), z) \leqslant m$, contradicting our assumption that $f(w') \in N_3$. ∎

It remains to show that $N_3 \neq \emptyset$. Since $|N_G(f(w))| \geqslant \lceil \frac{k}{t} \rceil$ and N_1, N_2, N_3 are pairwise disjoint, it suffices to show that $|N_1 \cup N_2| \leqslant \lceil \frac{k}{t} \rceil - 1$. Suppose $N_2 = \{a_1, ..., a_p\}$. By definition, for each $i \in [p]$, there exists some $a_i^* \in V(H)$ such that $dist_{G-f(w)}(a_i, a_i^*) \leqslant m$. Let $N_2^* = \{a_1^*, ..., a_p^*\}$. We claim that $a_1^*, ..., a_p^*$ are all distinct and $N_2^* \cap N_1 = \emptyset$. Suppose that $a_i^* = a_j^*$ for some $i \neq j$, then we would obtain a cycle of length at most $2m + 2 \leqslant 2t$ from the union of a shortest a_i, a_i^*-path in $G - f(w)$, a shortest a_j, a_j^*-path in $G - f(w)$, and the edges $a_i f(w)$ and $a_j f(w)$, a contradiction. Hence the a_i^*'s are all distinct. If $a_i^* \in N_1$ for some i, then again we can obtain a cycle of length at most $m + 2 \leqslant 2t$ from the union of a shortest a_i, a_i^*-path in $G - f(w)$ and the edges $a_i f(w)$, $a_i^* f(w)$, a contradiction. Hence $N_2^* \cap N_1 = \emptyset$.

Let $S = N_1 \cup N_2^*$. Then S is a subset of $V(H)$, and the above discussion shows that $|S| = |N_1 \cup N_2^*| = |N_1| + |N_2^*| = |N_1| + |N_2| = |N_1 \cup N_2|$. So it suffices to show that $|S| \leqslant \lceil \frac{k}{t} \rceil - 1$.

Claim 3. $|S| \leqslant \lceil \frac{k}{t} \rceil - 1$.

Proof. By Lemma 1, we may assume that T has a diameter of at least $2t+1$, in which case T'' has a diameter of at least $2t+1-(m+2) \geqslant t$ and hence contains at least t vertices. Since $T'' = L(A_2)$ omits vertices in $L(A_1)$ and w, and $L(A_1)$ contains at least $2m+1$ vertices by the proof of Claim 1, T'' contains at most $(k+1)-(2m+1)-1 = (k-1)-2m$ vertices. Hence H contains at most $(k-1)-2m$ vertices and at least t vertices. (Recall that $H = f(T'')$.)

Using girth type arguments as before, we have

$$\begin{aligned} dist_H(u, v) &> 2t-2 && \text{if} \quad u, v \in N_1, \quad u \neq v, \\ dist_H(u, v) &> 2t-2(m+1) && \text{if} \quad u, v \in N_2^*, \quad u \neq v, \end{aligned} \tag{2}$$

and

$$dist_H(u, v) > 2t-m-2 \qquad \text{if} \quad u \in N_1, \quad v \in N_2^*.$$

Consider the case $u \in N_1$, $v \in N_2^*$, for instance. If $dist_H(u, v) \leqslant 2t-m-2$, then the union of a shortest u, v-path in H, a shortest v, $f(w')$-path in $G - f(w)$, and the edges $f(w')\,f(w)$, $uf(w)$ contains a cycle of length at most $(2t-m-2)+m+2 = 2t$, a contradiction.

For each vertex $x \in S = N_1 \cup N_2^*$, define $B(x)$ as follows.

$$B(x) = \{\, y \in V(H) : dist_H(x, y) \leqslant t-1 \} \qquad \text{if} \quad x \in N_1, \tag{3}$$

and

$$B(x) = \{\, y \in V(H) : dist_H(x, y) \leqslant t-(m+1) \} \qquad \text{if} \quad x \in N_2^*.$$

By (2) and (3), we have $B(x) \cap B(x') = \varnothing$ for distinct vertices x, $x' \in S$.

Since H is connected and has at least t vertices, it is immediate from the definition that $|B(x)| \geqslant \min\{1+(t-1), n(H)\} = t$ if $x \in N_1$ and that $|B(x)| \geqslant \min\{1+[t-(m+1)], n(H)\} = t-m$ if $x \in N_2^*$.

We show that in fact for $x \in N_2^*$ we also have $|B(x)| \geqslant t$, with at most two exceptions. It will then follow that $(k-1)-2m \geqslant n(H) \geqslant |\bigcup_{x \in S} B(x)| = \sum_{x \in S} |B(x)| \geqslant (|S|-2)\,t + 2(t-m)$, from which we will get $|S| \leqslant \lfloor \frac{k-1}{t} \rfloor = \lceil \frac{k}{t} \rceil - 1$.

We consider two cases.

Case 1. $m=(t-1)/2$.

In this case, $t-(m+1)=m$, and we know from the proof of Claim 1 that A_2 consists of a path $v_0v_1 \cdots v_h$, where $v_h=w''$ is the unique neighbor of w in A_2, and a leaf u_i adjacent to v_i for each $i=1, ..., m-1$. Furthermore, each of $v_m, ..., v_h$ is a vertex of degree 2 in $D(T)$ and is adjacent to either none or at least two leaves in T.

Let F denote the subgraph of A_2 induced by $v_0, v_1, ..., v_{m-1}$ and $u_1, ..., u_{m-1}$. Let $F^*=L(F)$. It is easy to check that F^* (and hence $f(F^*)$) cannot contain three vertices with pairwise distance at least $m+2$. Because every pair in N_2^* has distance (in H) at least $2t-2(m+1)+1 \geqslant m+2$, we have $|N_2^* \cap V(f(F^*))| \leqslant 2$. So it suffices to show that $|B(x)| \geqslant t$ for $x \in N_2^* - V(f(F^*))$. Such a vertex x is either $f(v_j)$ or a leaf in H adjacent to $f(v_j)$, for some $j \geqslant m$. Since $dist_H(f(v_j), f(v_h)) \geqslant dist_H(x, f(v_h))-1 \geqslant 2t-m-2 > m$ (note that $x \in N_2^*$ and $f(v_h) \in N_1$), we also have $j<h-m$.

If $x=f(v_j)$, where $m \leqslant j < h-m$, then $f(v_{j-m}), ..., f(v_j), ..., f(v_{j+m})$ all belong to $B(x)$, since they have distance at most $m=t-(m+1)$ from x in H. Hence $|B(x)| \geqslant 2m+1=t$. If x is a leaf adjacent to $f(v_j)$, then by our earlier discussion x has a sibling leaf x' adjacent to $f(v_j)$. Now x, x', $f(v_{j-(m-1)}), ..., f(v_{j+(m-1)})$ are $2m+1=t$ vertices belonging to $B(x)$.

Case 2. $m<(t-1)/2$.

In this case, we have $t-(m+1) \geqslant m+1$. We prove that $|B(x)| \geqslant t$ for all $x \in N_2^*$. Recall that w'' is the unique neighbor of w in T''. Since $f(w'') \in N_1$ and $x \in N_2^*$, by (2) we have $dist_H(x, f(w'')) \geqslant 2t-m-1 > t-(m+1)$. Let $P=x_0x_1 \cdots x_{t-(m+1)}$ denote the initial portion of length $t-(m+1)$ on the unique x, $f(w'')$-path in H, where $x_0=x$. The $t-m$ vertices on P clearly belong to $B(x)$. Hence it suffices to show that $B(x)-P$ contains at least m vertices.

Since $x_1, ..., x_{t-(m+1)}$ have degree at least 2 in H, $f^{-1}(x_1), ..., f^{-1}(x_{t-(m+1)})$ belong to $D(T)$. Let $I=\{i \in [t-(m+1)] : f^{-1}(x_i)$ is a vertex of degree 2 in $D(T)\}$.

Let $j \in [t-(m+1)]-I$, then $f^{-1}(x_j)$ is either a leaf of $D(T)$ (which is possible only if $j=1$ and $f^{-1}(x_0)$ is a leaf of T) or a vertex of degree at least three in $D(T)$. In the former case, $f^{-1}(x_1)$ is adjacent to at least two leaves of T (Lemma 4). Hence x_1 is adjacent to at least one leaf z_1 of H which is different from x_0; z_1 is a neighbor of x_1 in H not on P. In the latter case, $f^{-1}(x_j)$ has a neighbor v_j in $D(T)$ which is not on $f^{-1}(P)$. If v_j is not a leaf of $D(T)$, then it has a neighbor v_j' in $D(T)$ not on $f^{-1}(P)$. If v_j is a leaf of $D(T)$, then it is adjacent to some leaf v_j' of T; clearly, v_j' is not on $f^{-1}(P)$. Let $z_j=f(v_j)$ and $z_j'=f(v_j')$. We have $z_j, z_j' \in V(H)-V(P)$, $dist_H(z_j, x)=dist_H(x_j, x_0)+1=j+1$, and $dist_H(z_j', x)=dist_H(x_j, x_0)+2 =j+2$.

Let $b = \min\{i : i \in I\}$. If $b \geqslant m+1$ or b doesn't exist (i.e., if $I = \emptyset$), then $[m] \cap I = \emptyset$. By our discussion above, for each $j \in [m]$, x_j has a neighbor z_j in H not on P which is at a distance $j+1 \leqslant m+1 \leqslant t-(m+1)$ from x. These z_j's are clearly distinct from each other. This yields $|B(x) - P| \geqslant m$ and we are done. We henceforth assume that $b \leqslant m$.

By our definition of b, $f^{-1}(x_b)$ is a vertex of degree of 2 in $D(T)$. Let A_b denote the component in $D(T) - f^{-1}(x_b)$ that doesn't contain w. Then A_b is contained in $A_2 \subseteq T''$ and does not contain $f^{-1}(x_{b+1}), ..., f^{-1}(x_{t-(m+1)})$. By Remark 1 (immediately after Claim 1), A_b has a depth at least m from $f^{-1}(x_b)$. Hence there exists a path in $A_b \cup f^{-1}(x_b)$ of length m from $f^{-1}(x_b)$ to a vertex in A_b. Such a path can clearly be extended to a path of length $m+1$ in $L(A_b) \cup f^{-1}(x_b) \subseteq T''$. The image of that path, denoted by Q, is path of length $m+1$ in H starting at x_b and avoiding $x_{b+1}, ..., x_{t-(m+1)}$. Note that Q has $m+2$ vertices.

Now, we consider two subcases.

Subcase 2.1. $b = 1$.

In this subcase, we have $V(P) \cap V(Q) = \{x_1\}$ or $\{x_0, x_1\}$, where $x_0 = x$. In either case, there are m vertices on $Q - P$ within distance $m+1 \leqslant t-(m+1)$ from x. Hence $B(x) - P$ contains at least m vertices, and we are done.

Subcase 2.2. $2 \leqslant b \leqslant m$.

By the definition of b, we have $x_1, ..., x_{b-1} \notin I$. By our earlier discussion, x_1 has a neighbor z_1 in H not on P, while for each $j \in \{2, ..., b-1\}$, x_j has a neighbor z_j in H not on P and z_j has a neighbor z'_j in H not on P; z_j and z'_j are at distance $j+1$ and $j+2$ from x, respectively. Since $j \leqslant b-1$, we have $j+2 \leqslant b+1 \leqslant m+1 \leqslant t-(m+1)$. Hence $z_1, z_2, z'_2, ..., z_{b-1}, z'_{b-1} \in B(x) - P$.

Let q, $0 \leqslant q \leqslant b$, denote the smallest index such that x_q lies on Q, then $P \cap Q = P[x_q, x_b]$ (see Fig. 1). Recall that Q has $m+2$ vertices. The first $(m+1) - \max\{q, b-q\}$ vertices on $Q - P$ are within distance $m+1 \leqslant t-(m+1)$ from $x_0 = x$ and hence belong to $B(x) - P$. If $q = 1$, then none of $z_2, z'_2, ..., z_{b-1}, z'_{b-1}$ lies on $Q - P$, so the first $(m+1) - (b-1)$ vertices on $Q - P$ together with $z_2, z'_2, ..., z_{b_1}, z'_{b-1}$ give $(m+1) - (b-1) + 2(b-2) = m+b-2 \geqslant m$ vertices in $B(x) - P$. If $q = 0$ or b, then none of $z_1, z_2, z'_2, ..., z_{b-1}, z'_{b-1}$ lies on $Q - P$. These $2b-3$ vertices together with the first $(m+1) - b$ vertices on $Q - P$ give $2b - 3 + (m+1) - b = m+b-2 \geqslant m$ vertices in $B(x) - P$. So we may assume that $2 \leqslant q \leqslant b-1$, which implies that $b \geqslant 3$. In this case, at most two of $z_1, z_2, z'_2, ..., z_{b-1}, z'_{b-1}$ (namely, z_q and z'_q) may lie on $Q - P$ (see Fig. 1). Now, $2b-5$ of those which are not on $Q - P$, together with the first $(m+1) - \max\{q, b-q\}$ vertices on $Q - P$, all

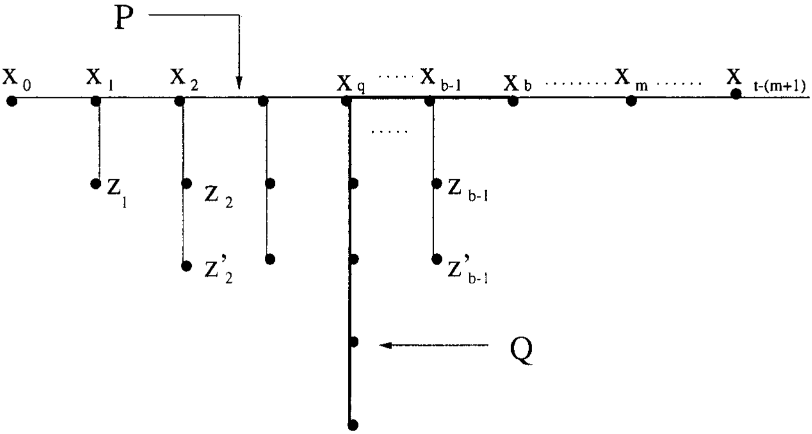

FIGURE 1

belong to $B(X)-P$. Hence, $|B(x)-P| \geqslant (2b-5)+(m+1)-\max\{q, b-q\}$. Since $(2b-5)+(m+1)-q=m+(b-3)+(b-1-q) \geqslant m$ and $(2b-5)+(m+1)-(b-q)=m+(b+q-4) \geqslant m+(3+2-4)>m$, we conclude that $|B(x)-P| \geqslant m$. This completes our proof. ▮

Note added. Since the initial submission of this paper, two papers on Dobson's conjecture have appeared. Haxell and Luczak [3] proved Dobson's conjecture for the case $t \geqslant 4$ and $k \geqslant 3t^2+2t$ and for the case $t \geqslant 3\lceil \log_{\lceil (k-1)/t \rceil} k \rceil \lceil \log_2 k \rceil$. See also [6].

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