

Vertex-Disjoint Cycles Containing Prescribed Vertices[†]

Yoshiyasu Ishigami¹ and Tao Jiang^{2*}

¹DEPARTMENT OF INFORMATION
AND COMMUNICATION ENGINEERING
THE UNIVERSITY OF ELECTRO-COMMUNICATIONS
TOKYO 182-8585, JAPAN
E-mail: yoshiyas@ice.uec.ac.jp

²DEPARTMENT OF MATHEMATICS AND STATISTICS
MIAMI UNIVERSITY
OXFORD, OHIO 45056
E-mail: jiangt@muohio.edu

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Abstract: Enomoto [7] conjectured that if the minimum degree of a graph G of order $n \geq 4k - 1$ is at least the integer $\lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1) + \frac{3}{2}k} - 1 \rfloor$, then for any k vertices, G contains k vertex-disjoint cycles each of

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*Correspondence to: Tao Jiang, Department of Mathematics and Statistics, Miami University, Oxford, OH 45056. E-mail: jiangt@muohio.edu

which contains one of the k specified vertices. We confirm the conjecture for $n \geq ck^2$ where c is a constant. Furthermore, we show that under the same condition the cycles can be chosen so that each has length at most six. © 2003 Wiley Periodicals, Inc. *J Graph Theory* 42: 276–296, 2003

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1. INTRODUCTION

All graphs considered in this paper are finite and simple. Given a graph $G = (V(G), E(G))$, we use $\delta(G)$ to denote the minimum degree of G , i.e., $\delta(G) = \min\{d(v) | v \in V(G)\}$ where $d(v) = |\{uv \in E(G) | u \in V(G)\}|$. Given a positive integer m , $[m]$ denotes the set $\{1, \dots, m\}$.

The study of conditions that guarantee a graph to contain a given number of vertex-disjoint or edge-disjoint cycles has a long history. Bollobás [2] devoted two sections of his book “Extremal Graph Theory” to this topic. One of the most elegant early results on this topic is the following one due to Corrádi and Hajnal [3].

Theorem 1.1 (Corrádi and Hajnal [3]). *Given $k \geq 1$, if the minimum degree of a graph G of order $n \geq 3k$ is at least $2k$ then G contains k vertex-disjoint cycles. This minimum degree condition is best possible.*

Recently, interest has grown in the study of degree conditions that ensure stronger properties. In proving a conjecture of Wang [11], Egawa et al. [5] investigated, for any given pair of integers n, k , the minimum integer $h(n, k)$ such that if G is an n -vertex graph with minimum degree $h(n, k)$ then for any k independent edges in G , there are k vertex-disjoint cycles in G , each of which contains exactly one of the k specified edges.

Theorem 1.2 (Egawa et al. [5]). *Let G be a graph of order $n \geq 4k - 1$, where $k \geq 2$ is an integer. If $\delta(G) \geq n/2 + k - 1$ then for any k independent edges e_1, \dots, e_k , G has k vertex-disjoint cycles C_1, \dots, C_k of length at most four such that $e_i \in E(C_i)$ for each $i \in [k]$.*

Furthermore, for infinitely many pairs n, k , there exists a graph H on n vertices with minimum degree $n/2 + k - 2$ and k independent edges in H such that there do not exist k vertex-disjoint cycles (of any length), each of which contains exactly one of the k edges.

In this paper, we study the vertex version of the problem. Given a set X of k vertices in a graph G , a *vertex-disjoint cycle cover*, or simply a *cycle cover* of X , is a collection of k vertex-disjoint cycles in G , each of which contains exactly one vertex in X . Given a pair of integers n and k , where $n \geq 3k$, let $f(n, k)$ denote the minimum integer m such that if G is a graph on n vertices with minimum degree m then given any set X of k vertices in G , G contains a cycle cover of X .

More generally, given integers n, k , and l , where $n \geq 3k$ and $3 \leq l \leq n$, we let $f_l(n, k)$ denote the minimum integer m such that if G is a graph on n vertices with

minimum degree m then for any set X of k vertices in G , G has a cycle cover \mathcal{C} of X such that each cycle in \mathcal{C} has length at most l . Note that $f_3(n, k) \geq f_4(n, k) \geq \dots \geq f_n(n, k)$, and that $f(n, k) = f_n(n, k)$. In [8], Ishigami studied $f_4(n, k)$.

Theorem 1.3 (Ishigami [8]). *Let $k \geq 1$ be an integer and G a graph of order $n \geq 3k$ with $\delta(G) \geq \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 1$. Then for any set X of k vertices, G has a cycle cover of X in which each cycle has length at most four.*

The degree condition is sharp. An example showing the sharpness is given in Ishigami [8] and also in Example 2 in Section 7.

In this paper, we determine $f(n, k)$ when n is large compared to k . For $n \geq ck^2$, where c is a large enough constant, we prove that $f(n, k) = f_6(n, k) = \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$. An example establishing $f(n, k) \geq \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$ is given by Example 4 in Section 7. For the main part of the paper, we prove that $f_6(n, k) \leq \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$. It then follows that $f(n, k) = f_n(n, k) = f_{n-1}(n, k) = \dots = f_6(n, k) = \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$. A detailed discussion about $f_l(n, k)$ for $l \in \{3, 4, \dots, n\}$ will be given in Section 7. Our main result is stated as follows, which was earlier conjectured to be true by Enomoto [6].

Theorem 1.4.1. *Let G be a graph with order $n \geq ck^2$, where c is a large enough absolute constant, and minimum degree at least $\lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$. Then for any k distinct vertices in G there exist k vertex-disjoint cycles of length at most six, each of which contains exactly one of the k specified vertices.*

We would like to point out that most of the effort in this paper is devoted to showing $f_6(n, k) \leq \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$ for $n \geq ck^2$. The proof would be simpler and the requirement $n \geq ck^2$ can be weakened if one only wishes to prove $f(n, k) \leq \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$.

For convenience, we prove the following theorem, which implies Theorem 1.4.1. Note that the condition $\delta \geq \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$ is equivalent to $n < \delta^2 + (-3k + 4)\delta - 2k + 3$ in our context.

Theorem 1.4.2. *Let δ, k be positive integers with $\delta \geq ck$, where c is a sufficiently large constant. Let G be a graph with order $n < \delta^2 + (-3k + 4)\delta - 2k + 3$ and minimum degree δ . Then given any set $X = \{x_1, \dots, x_k\}$ of k vertices in G , there exist k vertex-disjoint cycles C_1, \dots, C_k of length at most six, such that $x_i \in V(C_i)$ for each $i \in [k]$.*

Before we move on, we mention a related result and compare our result to that one. Given a set X of vertices in a graph G , if a cycle cover \mathcal{C} of X spans $V(G)$ (i.e. if \mathcal{C} forms a 2-factor of G), then we call it a *spanning cycle cover* of X . In [4], Egawa et al. studied the function $g(n, k)$, defined to be the minimum m such that if G is a graph on n vertices with minimum degree m then given any set X of k vertices in G , G contains a spanning cycle cover of X . Note that, trivially, $g(n, k) \geq f(n, k)$, while in reality $f(n, k)$ should be much smaller than $g(n, k)$. Egawa et al. [4] determined $g(n, k)$ for all feasible pairs n, k . In particular, for $n \geq 6k - 3$, it is shown that $g(n, k) = \lceil \frac{n}{2} \rceil$. As a key step in establishing the corresponding sharp upper bounds on $g(n, k)$, they first obtained upper bounds on $f_5(n, k)$, and then showed that a cycle cover of X in which each cycle has length at most five can be extended to a spanning cycle cover of X under the given degree conditions. While their upper bounds on $f_5(n, k)$ (Egawa et al. [4], Theorem 1.4.1) are good enough to enable them to obtain sharp upper bounds on $g(n, k)$, they are far too large to be sharp for $f_5(n, k)$. As we will see in Section 7, $f_5(n, k)$ is very close to $f_4(n, k)$, and is around $\sqrt{n} + ck$ for some small constant c , while the upper bound given in Egawa et al. [4] is roughly $n/2$ for large n . In general, studying $f(n, k)$ and studying $g(n, k)$ are two problems with quite different natures. To ensure spanning cycle covers, the host graph G often needs to be very dense (i.e., with its average degree on the order of $n = n(G)$). For dense graphs, especially for graphs G whose average degree exceed $n(G)/2$, Szemerédi's regularity lemma [13] has proven to be a very useful tool (see [10] for a survey and [9] for a recent application). On the other hand, the graphs we consider in this paper are sparse (having average degree as small as $O(\sqrt{n})$). No generally applicable powerful tools have yet been developed to handle sparse graphs effectively.

The rest of the paper is organized as follows. Sections 2–6 are devoted to the proof of Theorem 1.4.2. Section 7 contains a discussion of $f_l(n, k)$ for different values of l .

2. PRELIMINARIES

We introduce some notions needed for the proof. For undefined notations and terminology the reader is referred to [1, 12]. Let $G = (V(G), E(G))$ be a graph. Let $v \in V(G)$ and $U \subseteq V(G)$. Then $N(v, U)$ denotes the set $\{u \in U \mid uv \in E(G)\}$; it is the set of neighbors of v in U , and let $d(v, U) = |N(v, U)|$. Let $N[v, U] = N(v, U) \cup \{v\}$. When $U = V(G)$, we simply write $N(v), N[v]$ for $N(v, V(G))$ and $N[v, V(G)]$, respectively.

If U and W are two disjoint subsets of $V(G)$, then $E(U, W)$ denotes the set $\{uw \in E(G) \mid u \in U, w \in W\}$, i.e., the set of edges in G with one endpoint in U and the other endpoint in W . Let $e(U, W) = |E(U, W)|$. For convenience, we sometimes write $e(H, H'), e(v, H), N(v, H)$ for $e(V(H), V(H')), e(\{v\}, V(H)), N(\{v\}, V(H))$, respectively, where H and H' are subgraphs of G . The distance between two vertices u and v in a graph H is denoted by $dist_H(u, v)$. The subgraph of G induced by a subset S of $V(G)$ is denoted by $G[S]$. For the purpose of this paper,

we simply call a vertex-disjoint cycle cover of X in which each cycle has length at most six a *cover* of X .

Our approach to the proof of Theorem 1.1.2 is by contradiction. Let G be a graph with n vertices and minimum degree δ , where n and δ satisfy the inequality in Theorem 1.4.2, and let $X = \{x_1, \dots, x_k\}$ be a set of k vertices for which the desired cycles do not exist. We obtain a contradiction by deriving the inequality $n \geq \delta^2 + (-3k + 4)\delta - 2k + 3$.

We first deal with the case $k = 1$ separately, proving $n \geq \delta^2 + \delta + 1$ for this case. When $k = 1$, we have $X = \{x_1\}$ and that G has no cycle of length at most six containing x_1 . Let y_1, \dots, y_δ be distinct neighbors of x_1 . Clearly, $N(y_i) \cap N(y_j) = \{x_1\}$ for $i \neq j$; otherwise we obtain a cycle of length at most four containing x_1 . For each i , let $z_i \in N(y_i) - x_1$. We have $N(z_i) \cap N(z_j) = \emptyset$; otherwise we get a cycle of length at most six containing x_1 . Noting that $|N[z_i]| \geq \delta + 1$ for each i , we have $n = n(G) \geq 1 + \sum_i |N[z_i]| \geq 1 + \delta(\delta + 1) = \delta^2 + \delta + 1$. We henceforth assume that $k \geq 2$.

Since adding edges to G does not violate the conditions of G in Theorem 1.4.2, we may assume that G is *edge-maximal*, that is, G does not contain a cover of X but adding any non-edge to G yields a cover of X . In this case, it is easy to see that given any subset Y of X with order $k - 1$, G contains a cover of Y . Let Y be a subset of X with order $k - 1$, and let \mathcal{C} be a cover of Y . For any $y \in Y$, let C_y denote the cycle in \mathcal{C} that contains y . We define an *ear* to C_y as a path connecting y to a vertex on $C_y - \{y\}$ which is internally disjoint from $(\bigcup_{C \in \mathcal{C}} V(C)) \cup X$. For a vertex u in $V(G) - \bigcup_{C \in \mathcal{C}} V(C)$, a *pseudo ear* from u to C_y is a path from u to a vertex on C_y which is internally disjoint from $\bigcup_{C \in \mathcal{C}} V(C)$. The *ear number* of C_y is the maximum number of internally disjoint ears of length at most three that C_y has, and the ear number of \mathcal{C} is defined as the sum of the ear numbers of its cycles.

Now, among all the covers of a subset of X of order $k - 1$, we choose a cover \mathcal{C} such that

- (1) the total length of the cycles in \mathcal{C} is minimum, and
- (2) subject to (1) the ear number of \mathcal{C} is maximum.

Without loss of generality, we may assume that \mathcal{C} covers x_1, \dots, x_{k-1} . Let $\mathcal{I} = [k - 1]$, and let $\mathcal{C} = \{C_1, \dots, C_{k-1}\}$, where C_i covers x_i for all $i \in \mathcal{I}$. Let a_i, b_i denote the two neighbors of x_i on C_i for all $i \in \mathcal{I}$. Note that since G is edge-maximal, X induces a clique. Let $W = (\bigcup_{i \in \mathcal{I}} V(C_i)) \cup \{x_k\}$.

Lemma 2.1. *Let $u \in (V(G) - W) \cup \{x_k\}$, and let $i \in \mathcal{I}$. Then $|N(u, C_i)| \leq 3$. Furthermore, $|N(x_k, C_i)| = 3$ if and only if $N(x_k, C_i) = \{a_i, x_i, b_i\}$.*

Proof. If $u \in V(G) - W$ and u has four neighbors on C_i , then u has two neighbors v, v' on $C_i - x_i$ with distance at least three on $C_i - x_i$. In this case, we can replace the portion of $C_i - x_i$ between v and v' with vvv' to get a cycle C' containing x_i which is shorter than C_i , and $(\mathcal{C} - \{C_i\}) \cup \{C'\}$ forms a cover of $X - \{x_k\}$ with smaller total length than \mathcal{C} , a contradiction to our choice of \mathcal{C} .

Now, suppose $u = x_k$. Suppose that either $|N(x_k, C_i)| \geq 4$ or $|N(x_k, C_i)| = 3$ but $N(x_k, C_i) \neq \{a_i, x_i, b_i\}$. Then x_k has two neighbors v, v' on $C_i - x_i$ such that the portion L of C_i between v and v' which contains x_i has length at least three. In that case, we let $C' = (C_i - L) \cup vx_kv'$. Now, $(\mathcal{C} - \{C_i\}) \cup \{C'\}$ is a cover of $X - \{x_i\}$ with smaller total length than \mathcal{C} , again a contradiction to our choice of \mathcal{C} . ■

For $j = 1, 2, 3$, let $\mathcal{I}_j = \{i \in \mathcal{I} : e(x_k, C_i) = j\}$. Since X induces a clique, for $i \in \mathcal{I}_j$ and $j = 1, 2, 3$, x_k has $j - 1$ neighbors on $C_i - x_i$. By Lemma 2.1, $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ partition \mathcal{I} . For $i \in \mathcal{I}_2$, let v_i denote the unique neighbor of x_k on $C_i - x_i$. For $i \in \mathcal{I}_3$, Lemma 2.1 shows that a_i, b_i are the two neighbors of x_k on $C_i - x_i$.

Let $D = N(x_k, G - W)$. For each $y \in D$, let $S_y = N(y, G - W)$.

Lemma 2.2. *Let $y, y' \in D$, where $y \neq y'$, and let $z \in S_y \cup \{y\}$ and $z' \in S_{y'} \cup \{y'\}$. Then $\text{dist}_{G-W}(z, z') \geq 3$. In particular, this implies that $N[z, G - W] \cap N[z', G - W] = \emptyset$, and that $S_y \cap S_{y'} = \emptyset$.*

Proof. Otherwise, suppose that $\text{dist}_{G-W}(z, z') \leq 2$. Then $\text{dist}_{G-W}(y, y') \leq 4$, in which case, a shortest y, y' -path in $G - W$ together with yx_ky' forms a cycle of length at the most six containing x_k that is vertex disjoint from the cycles in \mathcal{C} . This cycle together with \mathcal{C} forms a cover of X , a contradiction. It follows that $N[z, G - W] \cap N[z', G - W] = \emptyset$. Letting $z = y$ and $z' = y'$ yields $S_y \cap S_{y'} = \emptyset$. ■

Next, we partition each of \mathcal{I}_1 and \mathcal{I}_2 into subsets as follows. Let

$$\begin{aligned} \mathcal{I}_1^A &= \{i \in \mathcal{I}_1 : \text{for some } y \in D, e(x_i, S_y) \geq 2\}, \\ \mathcal{I}_1^B &= \mathcal{I}_1 - \mathcal{I}_1^A, \\ \mathcal{I}_2^A &= \{i \in \mathcal{I}_2 : \text{for some } y \in D, e(x_i, S_y) \geq 2\}, \\ \mathcal{I}_2^B &= \{i \in \mathcal{I}_2 - \mathcal{I}_2^A : \text{there exists an ear of length at the most three to } C_i\}, \\ \mathcal{I}_2^C &= \mathcal{I}_2 - \mathcal{I}_2^A - \mathcal{I}_2^B. \end{aligned}$$

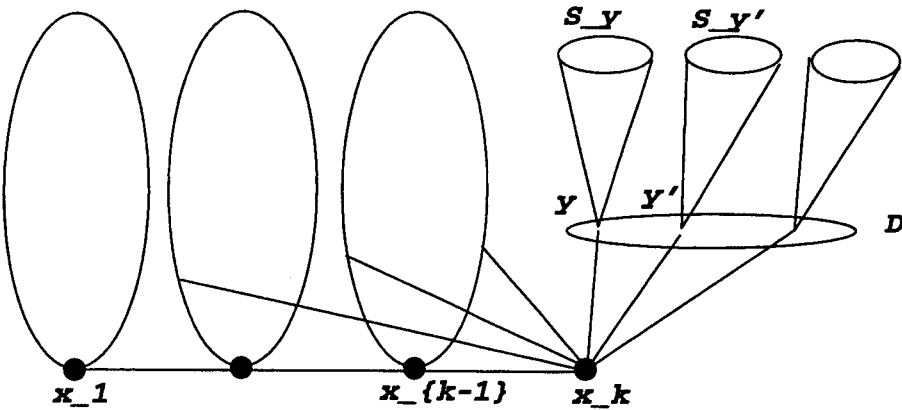


FIGURE 1. The main part of graph G.

Lemma 2.3. *For all $i \in \mathcal{I}_1^B \cup \mathcal{I}_2^B \cup \mathcal{I}_2^C \cup \mathcal{I}_3$ and all $y \in D$, x_i has at most one neighbor in S_y .*

Proof. For $i \in \mathcal{I}_1^B \cup \mathcal{I}_2^B \cup \mathcal{I}_2^C$, it follows from the definitions that x_i has at most one neighbor in S_y . Hence we may assume that $i \in \mathcal{I}_3$. Suppose that x_i has two neighbors u and u' in S_y . By Lemma 2.1, x_k is adjacent to the two neighbors a_i, b_i of x_i on C_i . Let C' denote the cycle obtained from C_i by replacing edges $a_i x_i b_i$ with $a_i x_k b_i$, and let $C'' = x_i u y u' x_i$. Then $(\mathcal{C} - \{C_i\}) \cup \{C', C''\}$ is a cover of X , contradicting our assumption. ■

Given $y \in D$, let $S_y^- = S_y - \bigcup_{i \in \mathcal{I}_1^B \cup \mathcal{I}_2^B \cup \mathcal{I}_2^C \cup \mathcal{I}_3} N(x_i)$.

Lemma 2.4. *$|D| \geq \delta - 3k + 3$. Also, for all $y \in D$, $|S_y| \geq \delta - 3k + 2$ and $|S_y^-| \geq \delta - 4k + 3$.*

Proof. Since x_k has at most three neighbors on C_i for each $i \in \mathcal{I}$, $|D| = |N(x_k, G - W)| \geq \delta(G) - 3(k - 1) \geq \delta - 3k + 3$. By Lemma 2.1, for each $i \in \mathcal{I}$, y has at most 3 neighbors on C_i . Hence y has at most $3(k - 1) + 1 = 3k - 2$ neighbors in W . It follows that $|S_y| = |N(y, G - W)| \geq \delta - 3k + 2$.

By Lemma 2.3, $|S_y^-| \geq |S_y| - |\mathcal{I}_1^B \cup \mathcal{I}_2^B \cup \mathcal{I}_2^C \cup \mathcal{I}_3| \geq |S_y| - (k - 1) \geq \delta - 4k + 3$. ■

Now, for each $y \in D$, we choose $z_y \in S_y^-$ such that $e(z_y, \bigcup_{i \in \mathcal{I}_2^C} C_i)$ is minimum. Let $Z = \{z_y : y \in D\}$. The following fact is clear from our definition of Z .

Fact 1. *For all $i \in \mathcal{I}_1^B \cup \mathcal{I}_2^B \cup \mathcal{I}_2^C \cup \mathcal{I}_3$, $e(x_i, Z) = 0$. Also, $|Z| = |D|$.*

Lemma 2.5. *Let $i \in \mathcal{I}$, and let P be an ear to C_i that intersects C_i at x_i and u . Then, each of the two portions of C_i between x_i and u has length at the most that of P .*

Proof. Otherwise, let L be the portion of C_i between x_i and u that is longer than P . Let $C' = (C_i - L) \cup P$. Now, C' is shorter than C_i and $(\mathcal{C} - \{C_i\}) \cup \{C'\}$ is a cover of $X - \{x_k\}$ with smaller total length than \mathcal{C} , contradicting our choice of \mathcal{C} . ■

In the following three sections, we prove a series of claims with the aim to obtain an upper bound on $e(Z, W)$. This upper bound on $e(Z, W)$, together with some other claims will be used in Section 6 to provide a lower bound on n , the number vertices in G , leading to the inequality that would give the contradiction.

3. THE CASE OF $i \in \mathcal{I}_1$

In this section, we get upper bounds of $e(Z, C_i)$ for $i \in \mathcal{I}_1$.

Claim 3.1. *For all $i \in \mathcal{I}_1^A$, $e(Z, C_i) \leq |D| + 6$.*

Proof. By the definition of \mathcal{I}_1^A , there exists $y \in D$ with $e(x_i, S_y) \geq 2$. Let u, u' be two neighbors of x_i in S_y . Then $C' = x_i u y u' x_i$ is a cycle of length four containing x_i , and $(\mathcal{C} - \{C_i\}) \cup \{C'\}$ is a cover of $X - \{x_k\}$. By our choice of \mathcal{C} , $|C_i| \leq |C'| = 4$.

Let v be an arbitrary vertex on $C_i - x_i$. We claim that $e(v, Z - \{z_y\}) \leq 1$. Otherwise, suppose that v has two neighbors $z_{y'}, z_{y''}$ in $Z - \{z_y\}$, where $y', y'' \in D$ are two vertices different from y . Let $C'' = v z_{y'} y' x_k y'' z_{y''} v$. C'' is a cycle of length six containing x_k , which is vertex disjoint from $(\mathcal{C} - C_i) \cup C'$. We thus obtain a cover of X , a contradiction. Hence $e(v, Z - \{z_y\}) \leq 1$ for all $v \in V(C_i) - \{x_i\}$.

Now, we have $e(Z - \{z_y\}, C_i) \leq |Z - \{z_y\}| + |C_i - x_i| \leq (|D| - 1) + 3$. Hence $e(Z, C_i) = e(z_y, C_i) + e(Z - \{z_y\}, C_i) \leq 4 + (|D| - 1) + 3 = |D| + 6$. ■

Claim 3.2. $\forall i \in \mathcal{I}_1^B, e(Z, C_i) \leq 2|D| + 4$.

Proof. By Fact 1, $e(Z, x_i) = 0$. Hence it suffices to show that $e(Z, C_i - x_i) \leq 2|D| + 4$. Let $z \in Z$. By the proof of Lemma 2.1, z has at most three neighbors on $C_i - x_i$, and if z has three neighbors on $C_i - x_i$ then they must be consecutive in order.

Call a portion of $C_i - x_i$ containing four consecutive vertices (or all of $V(C_i - x_i)$ if $C_i - x_i$ has fewer than four vertices) a *maximal portion*. Since $C_i - x_i$ has at most five vertices, $C_i - x_i$ has at most two maximal portions. Suppose that $e(Z, C_i - x_i) > 2|Z| + 4 = 2|D| + 4$, then there exist five vertices in Z with three neighbors on $C_i - x_i$. By the pigeonhole principle, three of them, say $z_{y_1}, z_{y_2}, z_{y_3}$, have their neighborhoods on $C_i - x_i$ in the same maximal portion. Also, by pigeonhole principle, two of the three, say z_{y_1} and z_{y_2} , have the same neighborhood on $C_i - x_i$.

Let u, u', u'' be the three consecutive vertices on $C_i - x_i$ adjacent to z_{y_1} and z_{y_2} . Since z_{y_3} 's three neighbors on $C_i - x_i$ belong to the same maximal portion as u, u', u'' , it is clear that u' is a neighbor of z_{y_3} . Let $C' = (C_i - \{uu', uu''\}) \cup \{uz_{y_1}, z_{y_1}u''\}$, and let $C'' = u'z_{y_2}y_2x_ky_3z_{y_3}u'$ (see Fig. 2). Now, C' and C'' are

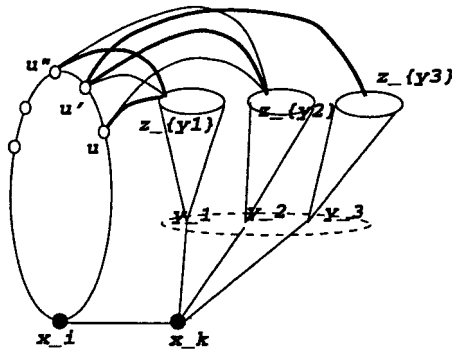


FIGURE 2. Proof of Claim 3.2.

vertex-disjoint cycles of length at the most six containing x_i and x_k , respectively, and $(\mathcal{C} - \{C_i\}) \cup \{C', C''\}$ is a cover of X , a contradiction. ■

Since $|D| \geq \delta - 3k + 3 \geq 2$, the last two claims yield the following.

Corollary 3.1. $e(Z, \bigcup_{i \in \mathcal{I}_1} C_i) \leq (2|D| + 4) \cdot |\mathcal{I}_1|$.

4. THE CASE OF $i \in \mathcal{I}_2$

In this section, we obtain upper bounds on $e(Z, C_i)$ for $i \in \mathcal{I}_2$.

Claim 4.1. For all $i \in \mathcal{I}_2^A, e(Z, C_i) \leq |D| + 3$.

Proof. By the definition of \mathcal{I}_2^A , there exists $y \in D$ such that $e(x_i, S_y) \geq 2$. Let u, u' be two neighbors of y in S_y , and let $C' = x_i u y u' x_i$. Since C' is a cycle of length four containing x_i , and $(\mathcal{C} - \{C_i\}) \cup \{C'\}$ is a cover of $X - \{x_k\}$, our choice of \mathcal{C} implies that $|C_i| \leq |C'| = 4$.

Now, if $e(Z - \{z_y\}, V(C_i) - \{x_i\}) = 0$ then we have $e(Z, C_i) = e(z_y, C_i) + e(Z - \{z_y\}, C_i) \leq |C_i| + (|Z| - 1) \leq 4 + |D| - 1 = |D| + 3$ and we are done. So, we may assume that $w z_{y'} \in E(G)$, for some $w \in V(C_i) - x_i$ and $y' \in D - \{y\}$. Recall that v_i denotes the unique neighbor of x_k on $C_i - x_i$. Since C_i has length at most four, the length of the portion of $C_i - x_i$ between w and v_i is at most two. This portion together with $w z_{y'} y' x_k v_i$ forms a cycle C'' of length at most six containing x_k (see Fig. 3a). Now, $(\mathcal{C} - \{C_i\}) \cup \{C', C''\}$ is a cover of X , which is a contradiction. ■

Claim 4.2. For all $i \in \mathcal{I}_2^B, e(Z, C_i) \leq |D| + 4$.

Proof. Recall that v_i denotes the unique neighbor of x_k on $C_i - x_i$ and that x_i has no neighbor in Z by our choice of Z (see Fact 1). Let $A = N(Z, C_i)$, then $A \subseteq V(C_i) - \{x_i\}$.

Suppose $|A| \geq 4$. Then $|C_i| \geq 5$ and A has a vertex b within distance $|C_i| - 5$ from v_i on $C_i - x_i$. Note that b is connected to x_k by a path Q of length three in $(G - W) \cup \{b\}$. Now, Q together with the edge $x_k v_i$ and the portion of C_i between v_i and b completes a cycle C^* of length at the most $(|C_i| - 5) + 3 + 1 = |C_i| - 1$ containing x_k . In that case, $(\mathcal{C} - \{C_i\}) \cup \{C^*\}$ is a cover of $X - \{x_{k-1}\}$ with smaller total length than \mathcal{C} , contradicting our choice of \mathcal{C} (see Fig. 3b). Hence $|A| \leq 3$.

By our definition of \mathcal{I}_2^B , there exists an ear L of length at most three to C_i . If there exist $y', y'' \in D, y' \neq y''$, such that L intersects $S_{y'} \cup y'$ at a vertex u' and intersects $S_{y''} \cup y''$ at a vertex u'' then since $u', u'' \in V(L) - \{u_i, x_i\}$, we have $dist_{G-W}(u', u'') \leq 1$, contradicting Lemma 2.2. Hence there is at most one vertex $y \in D$ such that L intersects $S_y \cup y$. Let $Z^- = Z - \{z_y : L \text{ intersects } S_y \cup \{y\}\}$. Then $|Z^-| \geq |Z| - 1$.

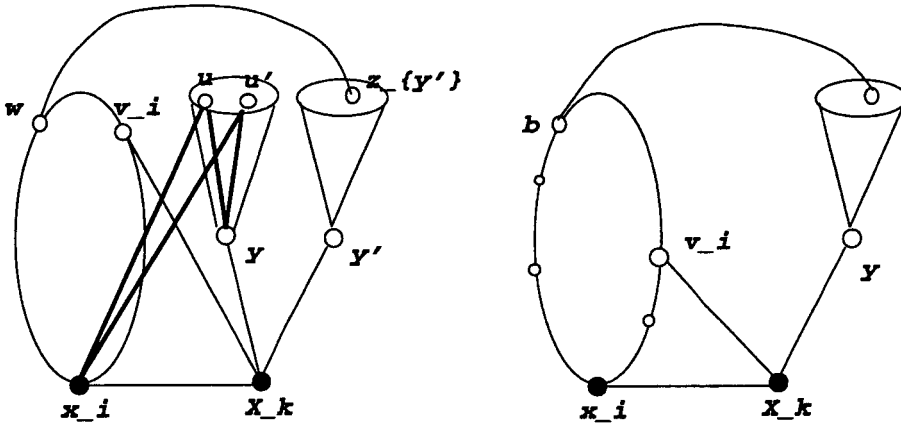


FIGURE 3. a: Proof of Claim 4.1 and (b) Proof of Claim 4.2.

Let u_i denote the other endpoint of L . By Lemma 2.5, each of the two portions of C_i between u_i and x_i has length at most three. Let $v \in C_i - \{x_i, u_i\}$, and let P be the portion of C_i between u_i and x_i that contains v . Suppose that v has two neighbors, say z_{y_1} and z_{y_2} , in Z^- . Let C' denote the cycle obtained from C_i by replacing P with L . C' is a cycle of length at most six containing x_i . Let $C'' = vz_{y_1}y_1x_ky_2z_{y_2}v$. C'' is a cycle of length six containing x_k . Now, $(C - \{C_i\}) \cup \{C', C''\}$ is a cover of X , a contradiction. Hence v has at most one neighbor in Z^- ; thus it has at most two neighbors in Z .

Recall that $|A| \leq 3$. If $u_i \notin A$, then our above discussion shows that $e(Z, C_i) = e(Z, A) \leq 2|A| \leq 6 \leq |D| + 4$. If $u_i \in A$, then $e(Z, C_i) = e(Z, A) = e(Z, u_i) + e(Z, A - u_i) \leq |D| + 2|A - u_i| \leq |D| + 4$. ■

Claim 4.3. *If $\delta \geq ck$, where c is a sufficiently large constant, then $e(Z, \bigcup_{i \in \mathcal{I}_2^c} C_i) \leq ck|\mathcal{I}_2^c|$.*

Proof. Let $i \in \mathcal{I}_2^c$. By the definition of \mathcal{I}_2^c , C_i has no ear of length at most three. Let $D_i = N(x_i, G - W)$. We first argue that we may assume $N(u, G - W) \cap N(u', G - W) \neq \emptyset$ if $u, u' \in D_i$ and $u \neq u'$. Otherwise, let $v \in N(u, G - W) \cap N(u', G - W)$, and let $C' = x_iuvv'u'x_i$. If $|C_i| \geq 5$, then replacing C_i with C' in \mathcal{C} yields a cover of $X - \{x_k\}$ with smaller total length than \mathcal{C} , a contradiction. Hence $|C_i| \leq 4$. Now, we may assume that there exist $y, y' \in D$ such that each of $z_y, z_{y'}$ has a neighbor on $C_i - x_i$, since otherwise, $|N(C_i - x_i, Z)| \leq 1$, in which case, our choice of Z implies $e(Z, C_i) \leq |C_i| - 1 \leq 3$, and it suffices to prove the claim for $\mathcal{I}_2^c - \{i\}$.

By Lemma 2.2, one of $\{y, z_y\}$ and $\{y', z_{y'}\}$ is disjoint from $V(C')$. Without loss of generality, we assume that $\{y, z_y\} \cap V(C') = \emptyset$. Let b denote a neighbor of z_y on $C_i - x_i$. The portion of $C_i - x_i$ between b and v_i , the unique neighbor of x_k on $C_i - x_i$, has length at most two, which together with $v_ix_kyz_yb$ completes a

cycle C'' of length at most six. Now, $(\mathcal{C} - \{C_i\}) \cup \{C', C''\}$ is a cover of X , a contradiction. Hence $N(u, G - W) \cap N(u', G - W) = \emptyset$ for all $u, u' \in D_i, u \neq u'$.

Since x_i has at most six neighbors on C_j for $j \in \mathcal{I} - \{i\}$ and two neighbors on C_i , we have $|D_i| \geq \delta - 6(k - 2) - 2 - 1 = \delta - 6k + 9$. Let $F_i = \bigcup_{u \in D_i} N[u, G - W]$. By Lemma 2.1, for all $u \in D_i, |N[u, G - W]| \geq 1 + (\delta - 3(k - 1) - 1) = \delta - 3k + 3$. By our discussions above, $|F_i| \geq |D_i| \cdot (\delta - 3k + 3) \geq (\delta - 6k + 9)(\delta - 3k + 3)$.

Since C_i has no ear of length at most three, we have $e(F_i, C_i - x_i) = 0$. Also, given $y \in D$, since $i \in \mathcal{I}_3^C$, by definition S_y^- does not contain a neighbor of x_i . Let $S^- = \bigcup_{y \in D} S_y^-$. Then $e(S^-, x_i) = 0$.

We have

$$\begin{aligned} e(S^-, C_i) &= e(S^- - F_i, C_i - x_i) \\ &\leq 5|S^- - F_i| \\ &\leq 5|V(G) - F_i| \\ &= 5(|V(G)| - |F_i|) \\ &\leq 5\{\delta^2 + (-3k + 4)\delta - 2k + 3\} - (\delta - 6k + 9)(\delta - 3k + 3) \\ &< 30k\delta. \end{aligned}$$

Hence, $e(S^-, \bigcup_{i \in \mathcal{I}_2^C} C_i) \leq 30k\delta \cdot |\mathcal{I}_2^C|$.

By our choice of z_y and Lemma 2.4, we have

$$\begin{aligned} e\left(S^-, \bigcup_{i \in \mathcal{I}_2^C} C_i\right) &= \sum_{y \in D} e\left(S_y^-, \bigcup_{i \in \mathcal{I}_2^C} C_i\right) \\ &\geq \sum_{y \in D} e\left(z_y, \bigcup_{i \in \mathcal{I}_2^C} C_i\right) \cdot |S_y^-| \\ &\geq \sum_{y \in D} e\left(z_y, \bigcup_{i \in \mathcal{I}_2^C} C_i\right) \cdot (\delta - 4k + 3) \\ &= e\left(Z, \bigcup_{i \in \mathcal{I}_2^C} C_i\right) \cdot (\delta - 4k + 3). \end{aligned}$$

Therefore, $e(Z, \bigcup_{i \in \mathcal{I}_2^C} C_i) \leq \frac{30k\delta}{(\delta - 4k + 3)} |\mathcal{I}_2^C| \leq 34k |\mathcal{I}_2^C|$ if $\delta \geq 34k$. ■

Under the assumption that $\delta \geq ck$, where c is a large constant, the last three claims together with the first statement of Lemma 2.4 yield:

Corollary 4.1. $e(Z, \bigcup_{i \in \mathcal{I}_2} C_i) \leq (|D| + 4) \cdot |\mathcal{I}_2|$.

5. THE CASE OF $i \in \mathcal{I}_3$

Recall that, for each $i \in \mathcal{I}_3$, a_i, b_i denote the two neighbors of x_i on C_i and that by Lemma 2.1 $N(x_k, C_i) = \{a_i, x_i, b_i\}$. Let

$$\mathcal{I}_3^A = \{i \in \mathcal{I}_3 : |N(C_i, Z)| \leq 1\},$$

and

$$\mathcal{I}_3^B = \{i \in \mathcal{I}_3 : |N(C_i, Z)| \leq 2\}.$$

Then \mathcal{I}_3^A and \mathcal{I}_3^B partition \mathcal{I}_3 . The following claim is immediate from our definitions of \mathcal{I}_3^A and Z .

Claim 5.1. *For all $i \in \mathcal{I}_3^A$, $e(C_i, Z) \leq |C_i| - 1$.* ■

For the case $i \in \mathcal{I}_3^B$, we prepare some lemmas.

Lemma 5.1. *Let $1 \leq i \leq k$. Let P_1, P_2 be two paths of length at most two such that $V(P_1) \cap V(P_2) = \{x_i\}$ and $V(P_j) \cap W = \{x_i\}$ for $j = 1, 2$. Let u be a vertex on $P_1 - x_i$ and v be a vertex on $P_2 - x_i$. Then $\text{dist}_{G-W}(u, v) \geq 3$.*

Proof. For $i = k$, we follow the proof of Lemma 2.2. For $i < k$, we switch the roles of x_i and x_k by replacing C_i with $(C_i - a_i x_i b_i) \cup a_i x_k b_i$, and apply similar arguments. ■

Lemma 5.2. *Let $i \in \mathcal{I}_3^B$. Let P be a pseudo ear from x_k to $C_i - x_i$ and Q be an ear to C_i . Suppose that P and Q are internally disjoint and that one of them has length at most three and the other has length at most four. Then $V(P) \cap V(C_i - x_i) = V(Q) \cap V(C_i - x_i)$.*

Proof. Suppose that $V(P) \cap V(C_i - x_i) = u$, $V(Q) \cap V(C_i - x_i) = v$, and $u \neq v$. Let L_1 denote the portion of C_i between u and x_i that avoids v , and let L_2 denote the portion of C_i between v and x_i that contains u . Without loss of generality, we may assume that L_1 contains a_i . Let L'_1 be the portion of L_1 between u and a_i . Let $C' = L_2 \cup Q$ and $C'' = x_k a_i \cup L'_1 \cup P$ (see Fig. 4); they are vertex-disjoint cycles that go through x_i and x_k , respectively. We show that both have length at most six, in which case $(C - C_i) \cup C' \cup C''$ forms a cover of X , a contradiction.

First, we assume that Q has length at most three and P has length at most four. In that case, Lemma 2.5 implies that each of the two portions of C_i between x_i and v has length at most three. It is straightforward to verify that C' and C'' both have length at most six.

Next, we assume that Q has length at most four and P has length at most three. If C'' is shorter than C_i then $(C - C_i) \cup C''$ is a cover of $X - x_i$ with smaller total length than C , a contradiction. Hence the portion of C_i between a_i and u that contains x_i has length at most length of P plus one, which is at most four (see

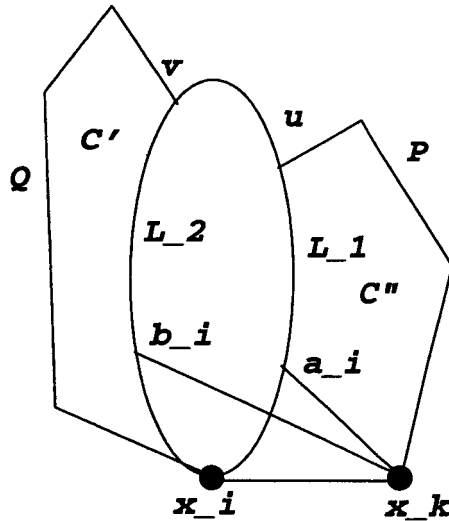


FIGURE 4. Proof of Lemma 5.2.

Fig. 4). This implies that L_2 has length at most two. So, C' has length at most six. Also, Lemma 2.5 implies that the portion of C_i between v and x_i that contains u has length at most four, hence L_1 has length at most two. Thus, C'' has length at most six. ■

Claim 5.2. For each $i \in \mathcal{I}_3^B$, there exists a vertex $w_i \in V(C_i) - \{x_i\}$ such that $N(Z, C_i - x_i) \subseteq \{w_i\}$. In particular, for all $i \in \mathcal{I}_3^B$, $e(Z, C_i) \leq |Z|$.

Proof. By our choice of Z , $e(x_i, Z) = 0$. Thus the first part of the claim implies the second part. Let $z_{y'}$ and $z_{y''}$ be two arbitrary vertices in Z that have neighbors on $C_i - x_i$. Let u be a neighbor of $z_{y'}$ on $C_i - x_i$, and let v be a neighbor of $z_{y''}$ on $C_i - x_i$ (see Fig. 5a). We prove that $u = v$, which will prove the first part of the claim.

Clearly, $P_1 = x_k y' z_{y'} u$ and $P_2 = x_k y'' z_{y''} v$ are two internally disjoint pseudo ears of length at most three from x_k to C_i . Let C^* be the cycle containing x_k obtained from C_i by replacing $a_i x_i b_i$ with $a_i x_k b_i$. Then P_1 and P_2 are two internally disjoint ears of length three to C^* . If C_i does not have two internally disjoint ears of length at most three, then $(C - \{C_i\}) \cup \{C^*\}$ would be a cover of $X - \{x_i\}$ with the same total length as C which has larger ear number than C , contradicting our choice of C . Hence, there exist two internally disjoint ears Q_1, Q_2 of length at most three to C_i . By Lemma 5.1, Q_1 must be internally disjoint from one of P_1 and P_2 . Assume that Q_1 is internally disjoint from P_1 . By Lemma 5.2, $V(Q_1) \cap V(C_i - x_i) = V(P_1) \cap V(C_i - x_i) = \{u\}$.

If P_2 is internally disjoint from Q_1 then Lemma 5.2 implies that $\{v\} = V(P_2) \cap V(C_i - x_i) = V(Q_1) \cap V(C_i - x_i) = \{u\}$. Hence, we may assume that P_2 intersects Q_1 internally at a vertex w . The union of the portion of Q_1 between

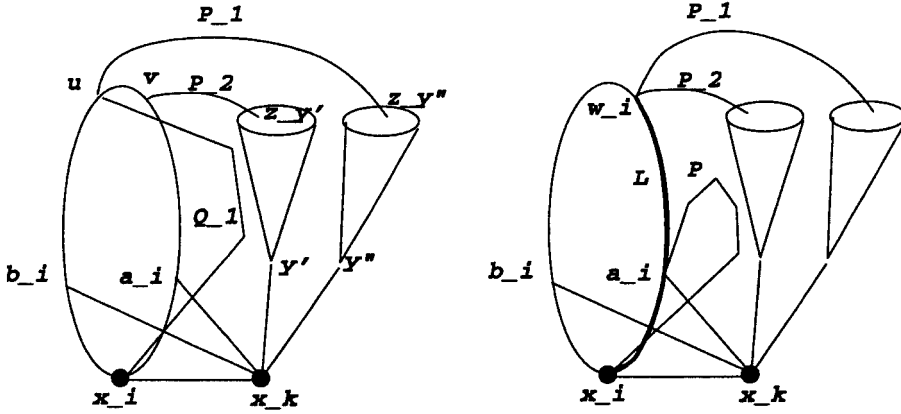


FIGURE 5. a: Proof of Claim 5.2 and (b) Proof of Lemma 5.3.

x_i and w and the portion of P_2 between w and v has length at most four and is internally disjoint from P_1 (since both Q_1 and P_2 are internally disjoint from P_1). This union contains an ear Q' of length at most four to C_i with endpoints x_i and v . Applying Lemma 5.2 to P_1 and Q' yields $u = v$. Hence there exists $w_i \in V(C_i - x_i)$ such that $N(Z, C_i - x_i) \subseteq \{w_i\}$. ■

For each $i \in \mathcal{I}_3^B$, either $a_i \neq w_i$ or $b_i \neq w_i$. By symmetry, we may henceforth assume that $a_i \neq w_i$, for all $i \in \mathcal{I}_3^B$.

Lemma 5.3. *Let $i \in \mathcal{I}_3^B$. Then*

- (i) *If Q is an ear or pseudo ear of length at most four to C_i then $V(Q) \cap V(C_i - x_i) = \{w_i\}$.*
- (ii) *If P is an ear of length at most five to C_i or a pseudo ear of length at most five from x_k to C_i , then $V(P) \cap V(C_i) \neq \{a_i\}$.*

Proof. (i) Let $z_{y'}$ and $z_{y''}$ be two arbitrary vertices in Z that have neighbors on $C_i - x_i$. Let $P_1 = x_k y' z_{y'} w_i$, and let $P_2 = x_k y'' z_{y''} w_i$. P_1 and P_2 are two internally disjoint pseudo ears from x_k to C_i . Let Q be an ear of length at most four to C_i . By Lemma 5.1, P_1 or P_2 , say P_1 , is internally disjoint from Q . By Lemma 5.2, $V(Q) \cap V(C_i - x_i) = V(P_1) \cap V(C_i - x_i) = \{w_i\}$. Similarly, if Q is a pseudo ear of length at most four, we consider Q and ears Q_1, Q_2 which were defined in the proof of Claim 5.2.

(ii) Let P be an ear of length at most five that intersects C_i at a_i . Suppose for now that P has length five and that $P = x_i u_1 u_2 u_3 u_4 a_i$ (see Fig. 5b). If P_1 or P_2 intersects $u_3 u_4 a_i$, then we obtain a pseudo ear of length at most four intersecting $C_i - x_i$ at $a_i \neq w_i$, contradicting (i) of the claim. Hence P_1, P_2 are vertex-disjoint from $u_3 u_4 a_i$. By Lemma 5.1, P_1 or P_2 , say P_1 , is also vertex-disjoint from $u_1 u_2$. Consequently, P_1 is vertex-disjoint from P . Let $C' = P \cup a_i x_i$,

and let $C'' = x_k b_i \cup L \cup P_1$, where L denotes the portion of $C_i - x_i$ between b_i and w_i . One can easily verify that C', C'' have length at most six. Now, $(C - C_i) \cup C' \cup C''$ forms a cover of X , a contradiction. The same arguments apply if the length of P is less than five.

In the case where P is a pseudo ear of length at most five from x_k to C_i , consider P and ears Q_1, Q_2 defined in Claim 5.2, and apply similar arguments as above. ■

For each $i \in \mathcal{I}_3^B$, let $R(a_i) = \{u \in V(G) - W : \text{dist}_{G-(W-\{a_i\})}(u, a_i) \leq 2\}$.

Claim 5.3. *All of the following statements hold.*

- (i) For all $i \in \mathcal{I}_3^B, |R(a_i)| \geq \delta - 3k + 3$.
- (ii) For all $i, j \in \mathcal{I}_3^B, i \neq j, R(a_i) \cap R(a_j) = \emptyset$.
- (iii) For all $i \in \mathcal{I}_3^B$ and $y \in D, R(a_i) \cap N[z_y, G - W] = \emptyset$.

Proof. (i) Let $x \in N(a_i, G - W)$, then $N[x, G - W] \subseteq R(a_i)$. By Lemma 2.1, x has at most three neighbors on each cycle in \mathcal{C} . Therefore, $|R(a_i)| \geq d(x, G - W) + 1 \geq \delta - 3(k - 1) - 1 + 1 = \delta - 3k + 3$.

(ii) Since $i, j \in \mathcal{I}_3^B$, by the proof of Claim 5.2, there are two internally disjoint ears Q_i, Q'_i of length at most three to C_i , and two internally disjoint ears Q_j, Q'_j of length at most three to C_j . Suppose $R(a_i) \cap R(a_j) \neq \emptyset$. Then there is a path L of length at most four connecting a_i and a_j which is internally disjoint from W . Furthermore, by Lemma 5.3 (ii), it is easy to see that $V(L) \cap V(Q_i \cup Q'_i \cup Q_j \cup Q'_j) = \emptyset$. Also, by Lemma 5.1, one of Q_j, Q'_j , say Q_j , is vertex-disjoint from Q_i . Consequently, Q_i, Q_j, L are pairwise vertex-disjoint.

Recall that Q_i intersects C_i at x_i and w_i . Similarly, Q_j intersects C_j at x_j and w_j . Let C'_i be the cycle containing x_i , obtained from C_i by replacing the portion of C_i between x_i and w_i that contains a_i with Q_i . Lemma 2.5 implies that C' has length at most six. Similarly, let C'_j be the cycle containing x_j , obtained from C_j by replacing the portion of C_j between x_j and w_j that contains a_i with Q_j ; C_j has length at most six. Finally, let $C'_k = a_j x_k a_i \cup L$; C'_k is a cycle of length at most six containing x_k . Now, $(C - \{C_i, C_j\}) \cup \{C'_i, C'_j, C'_k\}$ is a cover of X , a contradiction.

(iii) Suppose that there exists $y^* \in D$ such that $N[y^*, G - W] \cap R(a_i) \neq \emptyset$. Then there exists a pseudo ear of length at most five from x_k to C_i that intersects C_i at a_i . This contradicts Lemma 5.3 (ii). ■

6. LOWER BOUND ON THE NUMBER OF VERTICES

In this section, we first use Corollaries 3.1 and 4.1 and Claims 5.1 and 5.2 to obtain an upper bound on $e(Z, W)$. Then we use this upper bound on $e(Z, W)$ and Claim 5.3 to obtain a lower bound on n , showing that $n \geq \delta^2 + (-3k + 4)\delta - 2k + 3$, which will yield the desired contradiction and complete the proof of Theorem 1.4.2.

By Corollaries 3.1 and 4.1 and Claims 5.1 and 5.2, we have

$$\begin{aligned} e(Z, W) &\leq (2|D| + 4) \cdot |\mathcal{I}_1| + (|D| + 4) \cdot |\mathcal{I}_2| + \sum_{i \in \mathcal{I}_3^A} (|C_i| - 1) + |D| \cdot |\mathcal{I}_3^B| \\ &\leq |D|(2|\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3^B|) + 4|\mathcal{I}_1| + 4|\mathcal{I}_2| - |\mathcal{I}_3^A| + \sum_{i \in \mathcal{I}_3^A} |C_i|. \end{aligned}$$

Let $d = \delta - 3k + 3 + |\mathcal{I}_2| + 2|\mathcal{I}_1|$. By Lemma 2.2, we have

$$\begin{aligned} \left| \bigcup_{y \in D} N[z_y, G - W] \right| &\geq |D|(\delta + 1) - e(Z, W) \\ &\geq |D|(\delta + 1 - (2|\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3^B|)) - 4|\mathcal{I}_1| - 4|\mathcal{I}_2| \\ &\quad + |\mathcal{I}_3^A| - \sum_{i \in \mathcal{I}_3^A} |C_i| \\ &\geq (\delta - 3k + 3 + |\mathcal{I}_2| + 2|\mathcal{I}_1|)(\delta + 1) - (2|\mathcal{I}_1| + |\mathcal{I}_2|)d \\ &\quad - |\mathcal{I}_3^B|(\delta - 3k + 3 + |\mathcal{I}_2| + 2|\mathcal{I}_1|) - 4|\mathcal{I}_1| - 4|\mathcal{I}_2| \\ &\quad + |\mathcal{I}_3^A| - \sum_{i \in \mathcal{I}_3^A} |C_i| \\ &= (\delta^2 + (-3k + 4)\delta - 2k + 3) - k + (|\mathcal{I}_2| + 2|\mathcal{I}_1|)(\delta + 1) \\ &\quad - (2|\mathcal{I}_1| + |\mathcal{I}_2|)d - |\mathcal{I}_3^B|(\delta - 3k + 3 + |\mathcal{I}_2| + 2|\mathcal{I}_1|) \\ &\quad - 4|\mathcal{I}_1| - 4|\mathcal{I}_2| + |\mathcal{I}_3^A| - \sum_{i \in \mathcal{I}_3^A} |C_i| \\ &= (\delta^2 + (-3k + 4)\delta - 2k + 3) + 2|\mathcal{I}_1|(\delta - d - 1 - |\mathcal{I}_3^B|) \\ &\quad + |\mathcal{I}_2|(\delta - d - 3 - |\mathcal{I}_3^B|) - |\mathcal{I}_3^B|(\delta - 3k + 3) - k \\ &\quad + |\mathcal{I}_3^A| - \sum_{i \in \mathcal{I}_3^A} |C_i|. \end{aligned}$$

Claim 5.3 implies that $|\bigcup_{i \in \mathcal{I}_3^B} R(a_i)| \geq (\delta - 3k + 3) \cdot |\mathcal{I}_3^B|$, and that $\bigcup_{y \in D} N[z_y, G - W]$, $\bigcup_{i \in \mathcal{I}_3^B} R(a_i)$, and W are pairwise disjoint. Hence, we have

$$\begin{aligned} &n - (\delta^2 + (-3k + 4)\delta - 2k + 3) \\ &\geq \left| \bigcup_{y \in D} N[z_y, G - W] \right| + \left| \bigcup_{i \in \mathcal{I}_3^B} R(a_i) \right| + |W| \\ &\quad - (\delta^2 + (-3k + 4)\delta - 2k + 3) \\ &\geq 2|\mathcal{I}_1|(\delta - d - 1 - |\mathcal{I}_3^B|) + |\mathcal{I}_2|(\delta - d - 3 \\ &\quad - |\mathcal{I}_3^B|) - k + |\mathcal{I}_3^A| - \sum_{i \in \mathcal{I}_3^A} |C_i| + \left(\sum_i |C_i| + 1 \right) \end{aligned}$$

$$\begin{aligned}
 &\geq 2|\mathcal{I}_1|(3k - 3 - |\mathcal{I}_2| - 2|\mathcal{I}_1| - 1 - |\mathcal{I}_3^B|) \\
 &\quad + |\mathcal{I}_2|(3k - 3 - |\mathcal{I}_2| - 2|\mathcal{I}_1| - 3 - |\mathcal{I}_3^B|) \\
 &\quad - (|\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3^B|) + 3(|\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3^B|) \\
 &= 2|\mathcal{I}_1|(3k - 3 - 2|\mathcal{I}_1| - |\mathcal{I}_2| - |\mathcal{I}_3^B|) \\
 &\quad + |\mathcal{I}_2|(3k - 4 - 2|\mathcal{I}_1| - |\mathcal{I}_2| - |\mathcal{I}_3^B|) + 2|\mathcal{I}_3^B| \\
 &\geq 2|\mathcal{I}_1|(3k - 3 - 2(k - 1)) \\
 &\quad + |\mathcal{I}_2|(3k - 4 - 2(k - 1)) + 2|\mathcal{I}_3^B| \\
 &= 2|\mathcal{I}_1|(k - 1) + |\mathcal{I}_2|(k - 2) + 2|\mathcal{I}_3^B| \\
 &\geq 0,
 \end{aligned}$$

since $k \geq 2$. This yields $n \geq \delta^2 + (-3k + 4)\delta - 2k + 3$ and completes the proof of Theorem 1.4.2. ■

7. CYCLE COVERS WITH VARIOUS LENGTHS

Recall that $f_l(n, k)$ denotes the minimum integer m such that if G is an n -vertex graph with minimum degree m then for any set X of k vertices G contains a cycle cover of X using cycles of length at most l .

In this section, we consider $f_l(n, k)$ for each $l \in \{3, 4, \dots, n\}$. As we will see, each is either determined or almost determined. For convenience, we assume that n is sufficiently large compared to k in the following discussions.

For $l = 3$, it is not too hard to show that $f_3(n, k) \leq \lfloor \frac{n}{2} \rfloor + k$. The following example shows that $f_3(n, k) \geq \lfloor \frac{n}{2} \rfloor + k$, and hence we have $f_3(n, k) = \lfloor \frac{n}{2} \rfloor + k$.

Example 1. Let $d = \lfloor \frac{n}{2} \rfloor + k - 1$. For sufficiently large n with respect to k , we have $d \geq 3k - 1$. We construct an n -vertex graph H_3 with minimum degree d such that for some set X of k vertices in H_3 , H_3 does not have a cycle cover of X using triangles. Note that since $d \leq \frac{n}{2} + k - 1$, we have $n \geq 2d - 2k + 2$. Let X, A, B, D be vertex sets such that $|X| = k, |A| = k - 1, |D| = d - 2k + 2$, and $|B| = n - d - 1$. Consider a complete graph on $X \cup A \cup B \cup D$. The graph H_3 is obtained from this complete graph by deleting of the edges within D and all the edges between X and B . Consider any vertex $v \in V(H_3)$. It is straightforward to check that $d(v) = d$ if $v \in X$ and that $d(v) > d$ if $v \in A \cup D$. If $v \in B$, then $d(v) = n - 1 - k \geq 2d - 3k + 1 \geq d$ since $d \geq 3k - 1$. So H_3 has minimum degree d .

However, one can easily check that H_3 does not contain a cycle cover of X using triangles (see Fig. 6). ■

For $l = 4$, Theorem 1.3 asserts that $f_4(n, k) \leq \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 1$. We construct an example below to show that $f_4(n, k) \geq \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 1$, and hence equality holds in both inequalities.

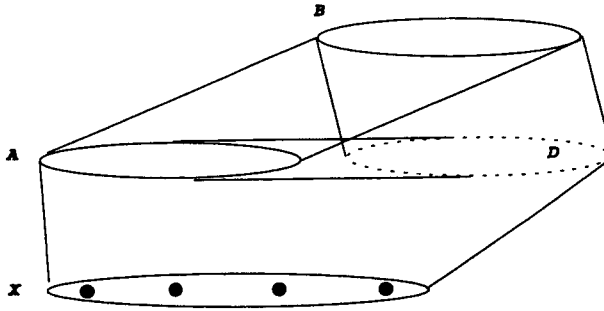


FIGURE 6. Graph H_3 .

Example 2. Let $d = \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 2$. For sufficiently large n with respect to k , we have $d \geq 3k$. Note that since $d \leq \sqrt{n + k^2 - 3k + 1} + 2k - 2$, we have $n \geq d^2 + (-4k + 4)d + 3k^2 - 5k + 3$. We construct an n -vertex graph H_4 with minimum degree d such that for some set X of k vertices, H_4 does not have a cycle cover of X using cycles of length at most four.

Let $X = \{x_1, \dots, x_k\}$, $A = \{a_1, \dots, a_{d-3k+2}\}$, $A', B, D_1, \dots, D_{d-3k+2}$ be pairwise disjoint sets such that $|X| = k$, $|A'| = 2k - 1$, $|A| = d - 3k + 2$, $|D_i| = d - k$, for each i , and $|B| = n - [d^2 + (-4k + 3)d + 3k^2 - 2k + 1]$. Since $n \geq d^2 + (-4k + 4)d + 3k^2 - 5k + 3$, we have $|B| \geq d - 3k + 2$.

Let $Z = B \cup (\cup_i D_i)$. Put a complete graph on each of the sets X , A' and Z , and put an empty graph on vertex set A . Then add all the edges between X and $A \cup A'$, and all the edges between A' and B . Finally, for each i , add all the edges between a_i and D_i . The resulting graph is the graph H_4 (see Fig. 7a). It is clear that H_4 has n vertices. Consider any vertex v in the graph. Clearly, if $v \in V(H_3) - Z$ then $d(v) \geq d$ with equality if $v \in X$. If $v \in Z$, then $d(v) \geq |Z| - 1 + 1 = |B| + \sum_i |D_i| \geq (d - 3k + 2) + (d - 3k + 2) \cdot (d - k) = d^2 + (-4k + 3)d + 3k^2 - 5k + 2$. Hence $d(v) - d \geq d^2 + (-4k + 2)d + 3k^2 - 5k + 2 \geq 0$ since $d \geq 3k$. Hence H_4 has minimum degree d . However, it is clear that H_4 does not contain a cycle cover of X using cycles of length at most four. ■

For $l = 5$, we did not make an effort to determine $f_5(n, k)$. It may not be very interesting, at least for large n , because the following example shows that $f_5(n, k) \geq \lfloor \sqrt{n + k^2 - 4k + 1} \rfloor + 2k - 1$. Since $f_5(n, k) \leq f_4(n, k) = \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 1$, we have $\lfloor \sqrt{n + k^2 - 4k + 1} \rfloor + 2k - 1 \leq f_5(n, k) \leq \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 1$. The lower and upper bounds do not differ by very much. For large n , they differ by at most one. In particular, we see that the difference between $f_5(n, k)$ and $f_4(n, k)$ is very small. For large n , $f_5(n, k)$ and $f_4(n, k)$ differ by at most one.

Example 3. Let $d = \lfloor \sqrt{n + k^2 - 4k + 1} \rfloor + 2k - 2$. Since $d \leq \sqrt{n + k^2 - 4k + 1} + 2k - 2$, we have $n \geq d^2 + (-4k + 4)d + 3k^2 - 4k + 3$.

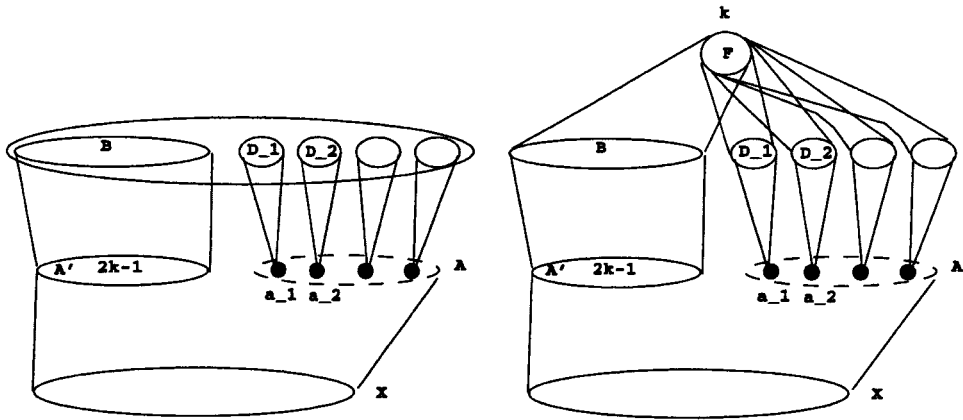


FIGURE 7. (a) Graph H_4 and (b) Graph H_5 .

We construct an n -vertex graph H_5 with minimum degree d such that for some set X of k vertices, H_5 does not have a cycle cover of X using cycles of length at most five. The construction of H_5 is very similar to that of H_4 .

Define vertex-disjoint sets $X, A, A', B, D_1, \dots, D_{d-3k+2}$ as in the construction of H_4 , where each set except B has the same order as before; let $|B| = n - [d^2 + (-4k + 3)d + 3k^2 - k + 1]$. Since $n \geq d^2 + (-4k + 4)d + 3k^2 - 4k + 3$, we have $|B| \geq d - 3k + 2$. Add another set F with k vertices. Let each of the sets X, A', B, F and the D_i 's induce a complete graph, and let A induce an independent set. Add all the edges between X and $A' \cup A$ and all the edges between F and $B \cup (\bigcup_i D_i)$. Finally, add all the edges between B and A' , and for each i , add all the edges between a_i and D_i . The resulting graph is H_5 (see Fig. 7b). One can verify that H_5 has n vertices and minimum degree d . However, H_5 has no cycle cover of X using cycles of length at most five. ■

The next example, suggested by H. Enomoto [6], shows that $f(n, k) \geq \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 1 \rfloor$.

Example 4. Let $d = \lfloor \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 2 \rfloor$. We construct an n -vertex graph H with minimum degree d such that for some set X of k vertices, H does not have a cycle cover of X .

Let $A, B_1, \dots, B_{d-3k+2}, X$ be disjoint sets of vertices where $|B_1| = \dots = |B_{d-3k+2}| = d + 1$, $|X| = k$, and $|A| = n - [d^2 + (-3k + 4)d - 2k + 3] + (d + 1)$. Since $d \leq \sqrt{n + (\frac{9}{4}k^2 - 4k + 1)} + \frac{3}{2}k - 2$, we have $|A| \geq d + 1$. Let C be a subset of A with order $2k - 1$. For each $i \in [d - 3k + 2]$, choose a vertex b_i in B_i . Let each of the sets $X, A, B_1, \dots, B_{d-3k+2}$ induce a complete graph. Then add all

the edges between the set X and the set $C \cup \{b_1, \dots, b_{d-3k+2}\}$. The resulting graph is the graph H . One can easily verify that H has n vertices and minimum degree d . However, H does not have a cycle cover of X , because in such a cycle cover each of the k cycles (which are pairwise vertex-disjoint) has to use at least two vertices from C , but C only has $2k - 1$ vertices. ■

Since $f_6(n, k) \geq \dots \geq f_n(n, k) = f(n, k)$, Theorem 1.4.1 and Example 4 together imply that $f_6(n, k) = \dots = f_n(n, k) = f(n, k) = \left\lfloor \sqrt{n + \left(\frac{9}{4}k^2 - 4k + 1\right) + \frac{3}{2}k} - 1 \right\rfloor$ for $n \geq ck^2$.

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