The rings of noncommutative projective geometry

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Abstract. In the past 15 years a study of "noncommutative projective geometry" has flourished. By using and generalizing techniques of commutative projective geometry, one can study certain noncommutative graded rings and obtain results for which no purely algebraic proof is known. For instance, noncommutative graded domains of quadratic growth, or "noncommutative curves," have now been classified by geometric data and these rings must be noetherian. Rings of cubic growth, or "noncommutative surfaces," are not yet classified, but a rich theory is currently forming. In this survey, we describe some of these results and examine the question of which rings should be included in noncommutative projective geometry.

1. Introduction

Noncommutative projective geometry is a new field of mathematics which studies noncommutative graded rings and their categories of modules via the techniques of projective algebraic geometry. This new "geometry" has been able to prove ring-theoretic theorems (for which no purely algebraic proof is known) and has led to new questions which a classical ring theorist would not have asked.

This paper will survey some of these new answers and questions. But first, we would like to say that an excellent survey article [**SV**] already exists. In fact, the lecture on which this paper is based was in turn based on that survey. Thus, similarities between the two papers must exist and we express our great debt to the previous article's authors. We hope that this simplified survey will encourage the reader to examine [**SV**]. However, we have also added some new material, particularly in Section 6.

To give this paper an overlying structure, we will give our opinionated answer to the question

Question 1.1. Which rings should be included in noncommutative projective geometry?

As mentioned before, we want our rings to be graded. More specifically, we usually assume

(R0) k is an algebraically closed field, $R = \bigoplus_{i=0}^{\infty} R_i$, $R_0 = k$, $\dim_k R_i < \infty$.

While some work [**AZ1, K1**] has dealt with the case where k is a commutative noetherian ring, most work has assumed that k is a field, often algebraically closed. A graded ring R is *finitely graded* when $\dim_k R_i < \infty$ and R is *connected* when $R_0 = k$. We will take *connected graded* to mean connected and finitely graded. We will sometimes weaken (R0) to just finitely graded, but in this paper, k is an algebraically closed field unless stated otherwise.

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It is clear that (R0) is much too broad for a successful use of geometric methods. The most obvious example is that of polynomials in two noncommuting variables, $k\langle x, y\rangle$. While this ring is finitely generated over k, it is not noetherian and even contains the subring $k\langle xy^i : i \in \mathbb{N}\rangle$, which is isomorphic to polynomials in countably many noncommuting variables. Since projective geometry relies heavily on the fact that finitely generated commutative k-algebras are noetherian, we stipulate that

(R1) R is right noetherian.

We shall continue to mark our cumulative answer to Question 1.1 by (R0), (R1), ... If an assumption (Rn) must be strengthened, we denote the new assumption (Rn'). We will base these assumptions on two guiding principles. First, that these properties hold for *twisted homogeneous coordinate rings*, the class of rings closest to commutative graded rings (in the sense of noncommutative projective geometry). Second, that these assumptions lead to useful generalizations of commutative geometry theorems.

An important specialization of Question 1.1 is

QUESTION 1.2. Which rings should generalize $k[x_0, \dots, x_n]$? In other words, which rings should be called a *noncommutative projective space* or *quantum* $\mathbb{P}^n$?

We will denote these extra assumptions as (QPn). We warn the reader that these answers (Rn) and (QPn) are simply the opinions of the author and should not be taken as an established answers to Questions 1.1 and 1.2. In particular, the definition of a quantum $\mathbb{P}^n$ changes from paper to paper, depending on the needs of each author.

2. Artin-Zhang Theorem

Noncommutative rings have few prime ideals compared to commutative rings. Hence, defining a projective scheme as a ringed topological space on the homogeneous primes of R would not be useful. Instead, we look to the Serre Correspondence Theorem (**SCT**) for a definition of noncommutative projective scheme.

First, we need some notation for module categories associated to R or a commutative scheme X.

- $\operatorname{Gr} R$ – the category of graded right R-modules,
- $\operatorname{Tors} R$ – the full subcategory generated by all $M \in \operatorname{Gr} R$ with $M_i = 0$ for all $i \gg 0$,
- $\operatorname{QGr} R$ – the quotient category $\operatorname{Gr} R / \operatorname{Tors} R$,
- $\operatorname{gr} R, \operatorname{tors} R, \operatorname{qgr} R$ – the respective subcategory of noetherian objects,
- $\operatorname{Qch} X, \operatorname{coh} X$ – the category of quasicoherent (resp. coherent) sheaves on the scheme X.

The SCT states that if R is commutative and finitely generated by R_1, then there is a category equivalence $\operatorname{QGr} R \cong \operatorname{Qch}(\operatorname{Proj} R)$. Further, under these equivalences, πR corresponds to $\mathcal{O}_X$, where πR is the image of R in $\operatorname{QGr} R$. Since the categories $\operatorname{QGr} R$ and $\operatorname{Qch} X$ are determined by their noetherian objects, the SCT is sometimes stated as the equivalence $\operatorname{qgr} R \cong \operatorname{coh} X$ [**H**, Exercise II.5.9].

We wish to generalize the SCT to other categories so that we may study $\operatorname{QGr} R$ when R is noncommutative. The SCT is proven by representing R as a homogeneous coordinate ring in high degree; we will define a "categorical" homogeneous coordinate ring. So let $\mathcal{C}$ be a locally noetherian k-linear abelian category (meaning that $\mathcal{C}$ is generated by its noetherian objects and that the objects and Hom-sets of $\mathcal{C}$ are k-vector spaces), let $\mathcal{O}$ be a noetherian object, and let s be an autoequivalence.

We define the homogeneous coordinate ring as

$$\Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0} = \bigoplus_{i=0}^{\infty} \operatorname{Hom}(\mathcal{O}, s^i \mathcal{O})$$

with multiplication given by the composition of homomorphisms

$$\operatorname{Hom}(\mathcal{O}, s^i\mathcal{O}) \times \operatorname{Hom}(\mathcal{O}, s^j\mathcal{O}) \cong \operatorname{Hom}(s^j\mathcal{O}, s^{i+j}\mathcal{O}) \times \operatorname{Hom}(\mathcal{O}, s^j\mathcal{O}) \\ \to \operatorname{Hom}(\mathcal{O}, s^{i+j}\mathcal{O}).$$

There are two categories and two autoequivalences which appear in the SCT. When $\mathcal{C} = \operatorname{QGr} R$ and $\mathcal{O} = \pi R$, there is a natural autoequivalence $(+1)$ given by $M \mapsto M(1)$ where $M(i)$ is the ith shift of a graded module M, that is, $M(i)_j = M_{i+j}$. (This is still an autoequivalence when R is noncommutative.) When X is a projective scheme with $\mathcal{C} = \operatorname{Qch} X, \mathcal{O} = \mathcal{O}_X$, and $s = - \otimes \mathcal{L}$ for an invertible sheaf $\mathcal{L}$, we have the homogeneous coordinate ring as it is normally defined.

Of course the proof of the SCT requires $\mathcal{L}$ to be ample, so we need a categorical definition of ampleness.

Definition 2.1. Let $\mathcal{C}$ be a locally noetherian k-linear abelian category, let $\mathcal{O} \in \mathcal{C}$ be a noetherian object, and let $s\colon \mathcal{C} \to \mathcal{C}$ be an autoequivalence. Write $s^i\mathcal{M} = \mathcal{M}(i)$. The pair $(\mathcal{O}, s)$ is *ample* if

(A1) For all noetherian $\mathcal{M} \in \mathcal{C}$, there exist $\ell_i > 0, i = 1, \ldots, p$, such that there exists an epimorphism

$$\bigoplus_{i=1}^{p} \mathcal{O}(-\ell_i) \twoheadrightarrow \mathcal{M},$$

(A2) For all epimorphisms $\mathcal{M} \twoheadrightarrow \mathcal{N}$, there exists n_0 such that

$$\operatorname{Hom}(\mathcal{O}, \mathcal{M}(n)) \to \operatorname{Hom}(\mathcal{O}, \mathcal{N}(n))$$

is an epimorphism for $n \geq n_0$.

Finally, we need a technical definition; we allow k to be any commutative noetherian ring, as we will need a more general definition for use in Section 6.

Definition 2.2. [**AZ1**, Definition 3.2, Proposition 3.1(3), 3.11] Let k be a commutative noetherian ring, let R be a right noetherian graded k-algebra where each R_i is a finite k-module. We say that R satisfies χ_j if for all $j' \leq j$, and all noetherian right graded R-modules M,

$$\bigoplus_{i=-\infty}^{\infty} \operatorname{Ext}^{j'}_{\operatorname{Gr} R}(R_0, M(i))$$

is a finite k-module. If R satisfies χ_j for all $j \geq 0$, we say that R satisfies χ.

We now return to assuming that k is an algebraically closed field. We may now state the Artin-Zhang Theorem, a noncommutative analogue of the SCT.

Theorem 2.3 (Artin-Zhang). [**AZ1**, Theorem 4.5] *Let $\mathcal{C}$ be a locally noetherian k-linear abelian category, let $\mathcal{O} \in \mathcal{C}$ be a noetherian object, and let $s\colon \mathcal{C} \to \mathcal{C}$ be an autoequivalence. Assume $\dim_k \operatorname{Hom}(\mathcal{M}, \mathcal{N}) < \infty$ for all noetherian $\mathcal{M}, \mathcal{N} \in \mathcal{C}$, and let $(\mathcal{O}, s)$ be ample. Set $B = \Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$. Then $\mathcal{C} \cong \operatorname{QGr} B$, and B is right noetherian and has χ_1.*

Conversely, if R is finitely graded, right noetherian, and satisfies χ_1, then we have that $R \cong \Gamma(\operatorname{QGr} R, \pi R, (+1))_{\geq 0}$ (in sufficiently high degree), $(\pi R, (+1))$ is ample, and $\dim_k \operatorname{Hom}(\mathcal{M}, \mathcal{N}) < \infty$ for all noetherian $\mathcal{M}, \mathcal{N} \in \operatorname{QGr} R$.

In light of the above theorem, we add to our list

(R2) R satisfies χ_1.

All commutative noetherian k-algebras with (R0) satisfy χ_1. Unfortunately, this is not the case for all noncommutative right noetherian rings R. Such rings can exhibit bad behavior; [**SZ**] present a family of rings which are (right or left) noetherian if and only if $\operatorname{char} k = 0$.

Definition 2.4. Let R satisfy (R0)–(R2). Then the pair $(\operatorname{QGr} R, \pi R)$ is a *noncommutative projective scheme.*

One may notice that in Definition 2.1, the property (A2) is weaker than the cohomological definition of an ample invertible sheaf given by the Serre Vanishing Theorem. For noncommutative projective schemes, this vanishing requires not just the condition χ_1, but χ.

Theorem 2.5. [**AZ1**, Corollary 7.5] *Let R satisfy* (R0)–(R2)*. Then R satisfies χ if and only if for all noetherian $\mathcal{M}$,*

1. $\dim_k \operatorname{Ext}^i(\pi R, \mathcal{M}) < \infty$ *for all* $i \geq 0$*, and*
2. *there exists n_0 such that* $\operatorname{Ext}^i(\pi R, \mathcal{M}(n)) = 0$ *for all* $n \geq n_0, i > 0$*.*

There are rings which satisfy χ_1 but not χ. We will say a bit more about one such example in the next section. We end this section with another definition concerning Ext.

Definition 2.6. Let $\mathcal{C}$ be a locally noetherian, k-linear category and let $\mathcal{O}$ be a noetherian object. The pair $(\mathcal{C}, \mathcal{O})$ has *finite cohomological dimension* n if n is minimal with respect to the property that $\operatorname{Ext}^i(\mathcal{O}, \mathcal{M}) = 0$ for all $\mathcal{M} \in \mathcal{C}$ and $i > n$. If R is a ring with (R0)–(R2) and $(\operatorname{QGr} R, \pi R)$ has finite cohomological dimension n, we say that R has finite cohomological dimension n.

3. Twisted homogeneous coordinate rings

In this section we will discuss those rings R which are closest to commutative rings in the sense of the Artin-Zhang Theorem. *Twisted homogeneous coordinate rings* are rings of the form

$$\Gamma(\operatorname{Qch} X, \mathcal{O}_X, s)_{\geq 0}$$

for some proper commutative scheme X, with $(\mathcal{O}_X, s)$ ample. By the Artin-Zhang Theorem, a ring R with (R0)–(R2) and $(\operatorname{QGr} R, \pi R) \cong (\operatorname{Qch} X, \mathcal{O}_X)$ for some commutative proper scheme X is isomorphic to a twisted homogeneous coordinate ring in sufficiently high degree.

Twisted homogeneous coordinate rings are so called because any autoequivalence of $\operatorname{Qch} X$ is of the form $\sigma_*(- \otimes \mathcal{L})$ for some automorphism σ and invertible sheaf $\mathcal{L}$ on X [**AZ1**, Corollary 6.9], [**AV**, Proposition 2.15]. Knowledge of automorphisms and invertible sheaves then allows one to study these twisted rings.

But before presenting some results, let us define a noncommutative analogue of Krull dimension, the *Gel'fand-Kirillov dimension*

$$\operatorname{GKdim} R = \inf\{\alpha \in \mathbb{R} \colon \dim_k \sum_{i=0}^{n} R_i \leq n^{\alpha} \text{ for all } n \gg 0\}.$$

If R is commutative, then $\operatorname{GKdim} R$ equals the Krull dimension of R [**KL**, Theorem 4.5]. It turns out that the GK-dimension can be 0, 1, any real number ≥ 2, or infinity [**KL**, Proposition 1.4, Theorems 2.5, 2.9]. It is an open question whether the GK-dimension of a graded domain must be in $\mathbb{N} \cup \{\infty\}$. Further, if R is a noetherian graded domain, must we have $\operatorname{GKdim} R \in \mathbb{N}$?

If GKdim $R = 0$, then R is a finite dimensional ring. If GKdim $R = 1$, then R is well-understood [**SSW**] and if R is a finitely generated domain, then R is commutative [**SW**]. One of the most interesting results of noncommutative projective geometry is the classification of domains with GK-dimension 2.

THEOREM 3.1. [**AS1**, Theorem 0.2] *Let R be a domain with* (R0)*, generated in degree one, and let* GKdim $R = 2$*. Then there exists a commutative projective curve Y and autoequivalence s of* Qch Y *such that $R \cong \Gamma(\operatorname{Qch} Y, \mathcal{O}_Y, s)_{\geq 0}$ (up to finite dimension) with $(\mathcal{O}_Y, s)$ ample. Thus R is right noetherian and has χ_1. (Actually, we will soon see that R is noetherian and has χ.)*

If R is a domain with GKdim $R = 2$, but no Veronese subring of R is generated in degree one, then R may be non-noetherian. It is also possible that R does not have χ_1.

Artin and Stafford extended their results to prime rings R and arbitrary fields k [**AS2**]. They obtain $\operatorname{qgr} R \cong \operatorname{coh} \mathcal{E}$, where $\mathcal{E}$ is a sheaf of orders on a projective curve Y inside a central simple $k(Y)$-algebra and coh $\mathcal{E}$ is the category of coherent $\mathcal{E}$-modules. For instance, with $Y = \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_Y + M_2(\mathcal{O}_Y(-1))$, one can generate an R which is a noetherian PI algebra which is not finite over its center [**AS2**, 0.2].

We now present important properties of any twisted homogeneous coordinate ring, which we believe other rings of noncommutative geometry should have.

THEOREM 3.2. [**K1, K2**] *Let R have* (R0)–(R2)*. Suppose that* $(\operatorname{QGr} R, \pi R) \cong (\operatorname{Qch} X, \mathcal{O}_X)$ *for some commutative scheme X which is proper over k. Then X is projective over k and,*

- (R1′) *R is noetherian,*
- (R2′) *R and R^{op} have χ (where R^{op} is the opposite ring),*
- (R3) *R and R^{op} have finite cohomological dimension,*
- (R4) *some Veronese subring $R^{(d)}$ is generated in degree one, and*
- (R5) GKdim R *is an integer.*

With the appropriate generalizations of definitions, if k is only assumed to be a commutative noetherian ring, then (R1′)–(R4) *are still true.*

In the theorem above, it is important that πR corresponds to $\mathcal{O}_X$ under the category equivalence. There exists a ring R with $(\operatorname{QGr} R, \pi R) \cong (\operatorname{Qch} \mathbb{P}^1, \mathcal{F})$ and $\mathcal{F} \neq \mathcal{O}_X$ such that R is right noetherian and has χ_1, but is not left noetherian and does not have χ [**AZ1**, Proposition 6.13].

We have already mentioned that a ring of GK-dimension 2 without (R4) may behave badly. The strengthened properties (R1′)–(R3) will be sufficient for an analogue of Serre Duality to hold. We will also see that (R1′)–(R5) are properties of "quantum polynomial rings," which we now examine.

4. Quantum polynomial rings

Since any commutative projective scheme can be embedded in $\mathbb{P}^n$ for some n, many projective geometry propositions are mainly concerned with projective n-space or its homogeneous coordinate ring $k[x_0, \ldots, x_n]$. Thus, we would like to define which rings R should generalize polynomial rings, then call R a *quantum polynomial ring* and call $(\operatorname{QGr} R, \pi R)$ a *noncommutative* $\mathbb{P}^n$. As of yet, there is no consensus on what this definition should be, so the reader should carefully examine the hypotheses of each paper dealing with noncommutative $\mathbb{P}^n$. We will explain some of the most common hypotheses.

First, since commutative polynomial rings are Gorenstein, we would like our quantum polynomials to satisfy a noncommutative analogue.

Definition 4.1. Let R be connected graded (R0). Then R is *Artin-Schelter Gorenstein (AS-Gorenstein)* if

(1) R has finite right and left injective dimension d, and
(2) For some shift ℓ, $\operatorname{Ext}^i_{\operatorname{Gr} R}(k, R) = \begin{cases} 0 & \text{if } i \neq d \\ k(\ell) & \text{if } i = d. \end{cases}$

Polynomial rings also have finite global dimension, so we make another definition.

Definition 4.2. Let R be connected graded (R0). Then R is an *Artin-Schelter regular (AS-regular) algebra* of dimension d if

(1) R has right and left global dimension d,
(2) $\operatorname{GKdim} R < \infty$, and
(3) R is AS-Gorenstein with injective dimension d.

Definition 4.3. Let R be connected graded (R0). Then R is a *quantum polynomial ring* of dimension d and $(\operatorname{QGr} R, \pi R)$ is a *noncommutative* $\mathbb{P}^d$ if

(QP1) R and R^{op} are AS-regular of dimension d,
(QP2) If ℓ is the shift in Definition 4.1, then $\ell = d$,
(QP3) R is a noetherian domain (so (R1′) holds),
(QP4) R is generated in degree one (so (R4) holds), and
(QP5) R has Hilbert series $(1-t)^{-d}$.

Our definition of quantum polynomial ring is stronger than that of many papers, but it has some nice consequences. Noetherian AS-Gorenstein rings satisfy (R2′) and (R3) [**YZ**]. Our hypotheses also imply that $\operatorname{GKdim} R$ is not only finite, but an integer [**Jg1**, Theorem 4.1], so R satisfies the (strengthened) (R0)–(R5).

Of course this definition is only good if it leads to useful results, and that it does. For instance, if R is a quantum polynomial ring which is also Cohen-Macaulay (in an appropriate sense), then the *Castelnuovo-Mumford regularity* of noetherian graded R-modules can be defined and has some of the same properties as commutative regularity. From this, the degrees of generators of some ideals can be bounded [**Jg1**]. There is also an *intersection theory* and *Bézout's theorem* on noncommutative $\mathbb{P}^n$ (with the condition that R is Auslander-Gorenstein, which is slightly stronger than AS-Gorenstein) [**MS**].

We should note that the condition (QP3) may be implied by some of the other (QPn). AS-regular algebras of dimension ≤ 3 have been classified and they are all noetherian domains [**ATV1, ATV2, St1, St2**]. (The classification of AS-regular algebras of dimension 3 was the original question which gave birth to noncommutative projective geometry.) If the Hilbert series does not equal $(1-t)^{-3}$, then the ring is considered a *weighted* quantum polynomial ring.

5. Serre duality

Perhaps the strongest tool which has carried over from commutative projective geometry is that of Serre duality. It has been instrumental in developing the intersection theory of [**Jg2, MS**]. It has also been used to analyze Gorenstein-like conditions; [**JZ**] gives generalizations of Watanabe's Theorem for AS-regular algebras and also Stanley's Theorem for "AS-Cohen-Macaulay" algebras.

Let us now define classical Serre duality. But first, we need a definition of "properness."

Definition 5.1. Let $\mathcal{C}$ be a noetherian k-linear abelian category. We call $\mathcal{C}$ Ext-*finite* if $\dim_k \operatorname{Ext}^i(\mathcal{M}, \mathcal{N}) < \infty$ for all i and all $\mathcal{M}, \mathcal{N} \in \mathcal{C}$. This can be thought of as saying that $\mathcal{C}$ is proper over k.

Definition 5.2. Let $\mathcal{C}$ be Ext-finite with distinguished object $\mathcal{O}$. Then $(\mathcal{C}, \mathcal{O})$ satisfies *classical Serre duality* if there exists $\omega^\circ \in D^b(\mathcal{C})$ such that $\mathrm{RHom}(-, \omega^\circ) \cong \mathrm{RHom}(\mathcal{O}, -)^*$ where * is the k-dual. If $(\mathcal{C}, \mathcal{O}) = (\operatorname{qgr} R, \pi R)$ for a connected graded ring R, we say that R satisfies classical Serre duality.

Of course this definition will only be useful if classical Serre duality holds for many $(\operatorname{qgr} R, \pi R)$. The condition on RHom forces R to have finite cohomological dimension, which we denoted as the good property (R3). Fortunately, the addition of some of our other properties is sufficient for classical Serre duality.

Theorem 5.3. [**YZ**] *Let R be connected graded* (R0) *and noetherian* (R1′). *Suppose*

(R2′) *R and R^{op} satisfy χ, and*

(R3) *R and R^{op} have finite cohomological dimension.*

Then R satisfies classical Serre duality.

Note in particular that our twisted homogeneous coordinate rings satisfy classical Serre duality, as do quantum polynomial rings. For quantum polynomial rings, the situation is even better, because when R satisfies (strengthened) (R0)–(R3) and is AS-Gorenstein of dimension $d+1$, then ω° is the image in $D^b(\operatorname{qgr} R)$ of an R-bimodule $\omega(d)$, left and right free of rank 1, and the duality statement becomes [**YZ**]

$$\mathrm{Ext}^i_{\operatorname{qgr} R}(-, \omega) \cong \mathrm{Ext}^{d-i}_{\operatorname{qgr} R}(\mathcal{O}, -)^*,$$

a familiar form of Serre duality for commutative Gorenstein varieties.

Before ending this section, we mention two other possible generalizations of Serre duality. First, Theorem 5.3 has a converse in some sense: If R is noetherian and connected graded, then $(\operatorname{qgr} R, \pi R)$ has a "balanced dualizing complex" if and only if R has (R2′) and (R3) [**V1, YZ**]. Second, a more category-theoretic version of Serre duality is given by a *Serre functor* $F\colon D^b(\mathcal{C}) \to D^b(\mathcal{C})$ which gives natural isomorphisms $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Hom}(\mathcal{B}, F\mathcal{A})^*$ for $\mathcal{A}, \mathcal{B} \in D^b(\mathcal{C})$ [**BK**].

6. Strongly noetherian rings and abstract Hilbert schemes

Changing the base of a projective scheme from k to another k-algebra is a powerful tool of algebraic geometry. There has been some recent work on base change in noncommutative geometry, which has led in turn to *abstract Hilbert schemes* which parameterize certain modules in $\operatorname{qgr} R$.

Of course to have a workable theory of base change, we need our rings to behave well under tensor products. We thus define

Definition 6.1. Let R be a k-algebra (not necessarily graded). Then R is *strongly right noetherian* if $R \otimes_k A$ is right noetherian for any commutative noetherian (not necessarily graded) k-algebra A.

Definition 6.2. Let R be a connected graded ring (R0). Then R satisfies the *strong χ* condition if $R \otimes_k A$ satisfies χ (as an A-algebra) for any commutative noetherian (not necessarily graded) k-algebra A.

(Our definition of strong χ is weaker than that in [**AZ2**, C6.8], but the statement of Theorem 6.3 will still be correct.)

Twisted homogeneous coordinate rings are strongly noetherian and they also satisfy strong χ [**ASZ**, Proposition 4.13] (while the second fact is not explicitly stated, it follows from the proof of the first fact). Thus we may wish to amend our list.

(R1″) R is strongly noetherian,

(R2″) R and R^{op} satisfy strong χ.

Unfortunately, not all noetherian k-algebras are strongly noetherian. [**RS**] gave a nongraded example. Recently, Jordan introduced a family of connected graded rings which are not strongly noetherian [**Jd**] and Rogalski showed that these rings are noetherian [**R**].

Besides giving a version of generic flatness for noncommutative rings [**ASZ**], the property of strongly noetherian also allows for the formation of Hilbert schemes.

Theorem 6.3. [**AZ2**, Theorem E5.1] *Let R be a connected graded, strongly noetherian ring, satisfying strong χ. Let P be a finitely graded, noetherian R-module. The quotients of πP in $\operatorname{qgr} R$ with fixed Hilbert function h are parameterized by a countable union of commutative projective schemes.*

Conjecture 6.4. [**AZ2**, Conjecture E5.2] *This union of schemes is actually a projective scheme.*

The algebraic space of Theorem 6.3 is directly analogous to the usual Hilbert scheme. However, [**AZ2**] also studies "Hilbert schemes" for abelian categories other than $\operatorname{qgr} R$. For instance, they obtain the following theorem. Note that in this case, the "Hilbert scheme" truly is a scheme.

Theorem 6.5. [**AZ2**, Theorem E4.3] *Let R be a connected graded, strongly noetherian ring. Let P be a finitely graded, noetherian R-module. The isomorphism classes of quotients of P in $\operatorname{gr} R$ with fixed Hilbert function h are parameterized by a commutative projective scheme.*

7. Category theoretic schemes

In Section 2, we defined a noncommutative projective scheme as $(\operatorname{QGr} R, \pi R)$ where R was a ring with some nice properties. Some work in noncommutative projective geometry is now moving into a new paradigm, that of considering any category with "geometric" attributes as a noncommutative space. It is felt that such a category should at least be a Grothendieck category in the following sense.

Definition 7.1. An abelian category $\mathcal{C}$ with generator and exact filtered direct limits is called a *Grothendieck category.* $\mathcal{C}$ automatically has products and enough injectives.

The following two definitions of a noncommutative surface are suggested in [**SV**]; we generalize the definition to that of a variety of dimension n.

Definition 7.2. (Attempt I) A *noncommutative normal projective variety* (possibly singular) of dimension n is a category $\mathcal{C}$ of the form $\operatorname{QGr} R$, where R is a noetherian connected graded domain with $\operatorname{GKdim} R = n + 1$ such that R is a maximal order (this is an analogue of normality). $\mathcal{C}$ is smooth if $\operatorname{QGr} R$ has homological dimension n (that is, n is minimal such that $\operatorname{Ext}^i(\mathcal{M}, \mathcal{N}) = 0$ for all $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ and all $i > n$).

Definition 7.3. (Attempt II) A *smooth noncommutative projective variety* of dimension n is a locally noetherian Grothendieck category $\mathcal{C}$ of homological dimension n such that the full subcategory of noetherian objects is Ext-finite and saturated. (Saturation differentiates between algebraic and analytic varieties; it turns out that $\operatorname{qgr} R$ is saturated when R and R^{op} are connected graded, noetherian, have χ and finite cohomological dimension, and $\operatorname{qgr} R$ has finite homological dimension [**SV**]).

It is not clear which, if either, definition is "correct." Work has been done on defining integral noncommutative schemes (that is, integral Grothendieck categories) [**Sm**], so this may help refine Definition 7.3.

For both of these definitions, only the dimension 1 case, the curve case, has been settled. The noncommutative curves in the sense of the first definition have been completely classified by [**AS1**] (even when R is not generated in degree one). Recently, curves in the sense of the second definition have also been classified, falling into only two different types. If one relaxes "saturation" to "having Serre duality via a Serre functor," then there are five different types [**RV**].

The surface case is a wide open problem for both definitions. It is interesting to note that the two surface definitions conflict slightly. More precisely, [**AV**] gives an example of a twisted homogeneous coordinate ring R which is obtained from a smooth elliptic surface X, yet $\operatorname{GKdim} R = 5$, not the expected 3, so $\operatorname{QGr} R$ is not a surface in the sense of Definition 7.2. However, since $\operatorname{QGr} R \cong \operatorname{Qch} X$, Definition 7.3 is satisfied.

There is now a theory of blowing up a point on a noncommutative smooth surface in the sense of Definition 7.3 [**V2**]. However, there is not yet a "Zariski's Main Theorem" or an accepted concept of "birational equivalence class," so a classification of surfaces seems a long way off. (See [**SV**, §10.1] for a discussion of possible definitions for birational equivalence.)

Therefore, much work has been done on specific examples for both definitions. We have already seen a definition in Section 4 for noncommutative $\mathbb{P}^2$ which is in line with Definition 7.2. Bondal and Polishchuk define more Grothendieck categories to be noncommutative $\mathbb{P}^2$'s using a construction called "$\mathbb{Z}$-algebras" (which are actually not rings in general).

THEOREM 7.4. [**BP**] *Any Bondal-Polishchuk noncommutative* $\mathbb{P}^2$ *"comes from" an AS-regular algebra R of dimension* 3 *with Hilbert series* $(1-t)^{-3}$. *That is, our original Definition 4.3 of noncommutative* $\mathbb{P}^2$ *and that of Bondal and Polishchuk are in some sense the same.*

A definition of noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ should include AS-regular algebras R of dimension 3 with Hilbert series $(1-t)^{-2}(1-t^2)^{-1}$ (that is, weight $(1,1,2)$). But if one makes a Bondal-Polishchuk type definition of $\mathbb{P}^1 \times \mathbb{P}^1$ via $\mathbb{Z}$-algebras, one does get new categories which don't come from such R. For more details on this, as well as other recent surface generalizations, such as $\mathbb{P}^1$-bundles, cubic surfaces, and Cremona transformations, we direct the reader to [**SV**] again.

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References

ASZ. M. Artin, L. W. Small, and J. J. Zhang, *Generic flatness for strongly Noetherian algebras*, J. Algebra **221** (1999), no. 2, 579–610. MR **2001a:**16006

AS1. M. Artin and J. T. Stafford, *Noncommutative graded domains with quadratic growth*, Invent. Math. **122** (1995), no. 2, 231–276. MR **96g:**16027

AS2. ______, *Semiprime graded algebras of dimension two*, J. Algebra **227** (2000), no. 1, 68–123. MR 1 754 226

ATV1. M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, Vol. I, Birkhäuser Boston, Boston, MA, 1990, pp. 33–85. MR **92e:**14002

ATV2. ______, *Modules over regular algebras of dimension* 3, Invent. Math. **106** (1991), no. 2, 335–388. MR **93e:**16055

AV. M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra **133** (1990), no. 2, 249–271. MR **91k:**14003

AZ1. M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), no. 2, 228–287. MR **96a:**14004

AZ2. ———, *Abstract Hilbert schemes*, Algebr. Represent. Theory **4** (2001), no. 4, 305–394. MR 1 863 391

BK. A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337. MR **91b:**14013

BP. A. I. Bondal and A. E. Polishchuk, *Homological properties of associative algebras: the method of helices*, Izv. Ross. Akad. Nauk Ser. Mat. **57** (1993), no. 2, 3–50. MR **94m:**16011

H. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math., no. 52, Springer-Verlag, New York, 1977. MR 57 #3116

Jd. D. A. Jordan, *The graded algebra generated by two Eulerian derivatives*, Algebr. Represent. Theory **4** (2001), no. 3, 249–275. MR 1 851 999

Jg1. P. Jørgensen, *Non-commutative Castelnuovo-Mumford regularity*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 2, 203–221. MR **2000h:**13010

Jg2. ———, *Intersection theory on non-commutative surfaces*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5817–5854. MR **2001b:**14005

JZ. P. Jørgensen and J. J. Zhang, *Gourmet's guide to Gorensteinness*, Adv. Math. **151** (2000), no. 2, 313–345. MR **2001d:**16023

K1. D. S. Keeler, *Ample filters of invertible sheaves*, arXiv:math.AG/0108068, J. Algebra, to appear, 2001.

K2. ———, *Criteria for σ-ampleness*, J. Amer. Math. Soc. **13** (2000), no. 3, 517–532. MR **2001d:**14003

KL. G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, revised ed., American Mathematical Society, Providence, RI, 2000. MR **2000j:**16035

MS. I. Mori and S. Paul Smith, *Bézout's theorem for non-commutative projective spaces*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 279–299. MR **2001m:**16075

RV. I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366. MR 1 887 637

RS. R. Resco and L. W. Small, *Affine Noetherian algebras and extensions of the base field*, Bull. London Math. Soc. **25** (1993), no. 6, 549–552. MR **94m:**16029

R. D. Rogalski, *Examples of generic noncommutative surfaces*, arXiv:math.RA/0203180, 2002.

SSW. L. W. Small, J. T. Stafford, and R. B. Warfield, Jr., *Affine algebras of Gel'fand-Kirillov dimension one are PI*, Math. Proc. Cambridge Philos. Soc. **97** (1985), no. 3, 407–414. MR **86g:**16025

SW. L. W. Small and R. B. Warfield, Jr., *Prime affine algebras of Gel'fand-Kirillov dimension one*, J. Algebra **91** (1984), no. 2, 386–389. MR **86h:**16006

Sm. S. P. Smith, *Integral non-commutative spaces*, J. Algebra **246** (2001), no. 2, 793–810. MR 1 872 125

SV. J. T. Stafford and M. Van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 2, 171–216. MR **2002d:**16036

SZ. J. T. Stafford and J. J. Zhang, *Examples in non-commutative projective geometry*, Math. Proc. Cambridge Philos. Soc. **116** (1994), no. 3, 415–433. MR **95h:**14001

St1. D. R. Stephenson, *Artin-Schelter regular algebras of global dimension three*, J. Algebra **183** (1996), no. 1, 55–73. MR **97h:**16053

St2. ———, *Algebras associated to elliptic curves*, Trans. Amer. Math. Soc. **349** (1997), no. 6, 2317–2340. MR **97m:**16080

V1. M. Van den Bergh, *Existence theorems for dualizing complexes over non-commutative graded and filtered rings*, J. Algebra **195** (1997), no. 2, 662–679. MR **99b:**16010

V2. ———, *Blowing up of non-commutative smooth surfaces*, Mem. Amer. Math. Soc. **154** (2001), no. 734, x+140. MR 1 846 352

YZ. A. Yekutieli and J. J. Zhang, *Serre duality for noncommutative projective schemes*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 697–707. MR **97e:**14003

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