Iterated elementary embeddings and the model theory of infinitary logic

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Abstract

We use iterations of elementary embeddings derived from the nonstationary ideal on ω_1 to reprove some classical results about the number of models of cardinality $\aleph_1$ in various infinitary logics. We also consider Galois stability in light of Burgess's theorem on analytic equivalence relations and find variants of the earlier theorems for Abstract Elementary Classes.

In this paper we use iterated generic elementary embeddings to analyze the number of models in $\aleph_1$ in various infinitary logics and for Abstract Elementary Classes. The arguments presented here are very much in the spirit of [7, 8], in which these embeddings were used to prove forcing-absoluteness results. Those papers focused on the large cardinal context. Here we work primarily in ZFC, though we note several cases where our results can be extended assuming the existence of large cardinals. The technique here provides a uniform method for approaching and extending problems that Keisler et al. proved in the 1970's.

We refer the reader to [1] for model-theoretic definitions such as *Abstract Elementary Class* and for background on the notions used here. For example, Theorem 0.2 is stated for atomic models of first order theories. The equivalence between this context and models of a complete sentence in $L_{\omega_1,\omega}$ is explained in Chapter 6 of [1]. Abstract Elementary Classes form a more general context unifying many of the properties of such infinitary logics as $L_{\omega_1,\omega}$, $L_{\omega_1,\omega}(Q)$, and $L_{\omega_1,\omega}(aa)$.

A fundamental result in the study of $\aleph_1$-categoricity for Abstract Elementary Classes is the following theorem of Shelah (see [1], Theorem 17.11).

Theorem 0.1 (Shelah)**.** *Suppose that* $\mathbf{K}$ *is an Abstract Elementary Class such that*

- $\mathrm{LS}(\mathbf{K}) = \aleph_0$*;*
- $\mathbf{K}$ *is* $\aleph_0$*-categorical;*
- *amalgamation fails for countable models in* $\mathbf{K}$[1]*.*

[1] Unlike first order logic, this is a strictly stronger statement than 'amalgamation fails over subsets of models of $\mathbf{K}$.'

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

Theorem 0.1 is one of the two fundamental tools to develop the stability theory of $L_{\omega_1,\omega}$. The second is the following theorem of Keisler (see [1], Theorem 18.15).

Theorem 0.2 (Keisler)**.** *Suppose that* $\mathbf{K}$ *is the class of atomic models of a complete first order theory, and that uncountably many types over the empty set are realized in some uncountable model in* $\mathbf{K}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

The notion of ω-stability for sentences in $L_{\omega_1,\omega}$ is a bit subtle and is more easily formulated for the associated class $\mathbf{K}$ of atomic models of a first theory. For *countable* $A \subseteq M \in \mathbf{K}$, $S_{at}(A)$ denotes the set of types over A realized in atomic models[2]. $\mathbf{K}$ is ω-stable if for each countable $M \in \mathbf{K}$, $|S_{at}(M)| = \aleph_0$ [3].

Combining these two theorems, Shelah showed (under the assumption $2^{\aleph_0} < 2^{\aleph_1}$) that a complete sentence of $L_{\omega_1,\omega}$ which has less that $2^{\aleph_1}$ models in $\aleph_1$ has the amalgamation property in $\aleph_0$ and is ω-stable. Crucially, the Shelah's argument relies on the assumption $2^{\aleph_0} < 2^{\aleph_1}$ in two ways. It first uses a variation of the Devlin-Shelah weak diamond principle [5] for Theorem 0.1. Then using amalgamation, extending Keisler's theorem from types over the empty set to types over a countable model is a straightforward counting argument, as it is in this paper. We work on analogs of this analysis for arbitrary AEC in Section 4.

Using the iterated ultrapower approach we give a new proof of an extension of Theorem 0.2 to the logic $L_{\omega_1,\omega}(\mathrm{aa})$ (as claimed in [16]). Again, it suffices to consider the case where amalgamation holds. Theorem 0.3 follows from Theorem 2.4 below.

Theorem 0.3. *Suppose that* $\mathbf{K}$ *is the class of models of some fixed sentence of* $L_{\omega_1,\omega}(\mathrm{aa})$*, and that, for some countable set* F *of* $L_{\omega_1,\omega}(\mathrm{aa})$*-sentences, uncountably many* F*-types are realized over some countable model in* $\mathbf{K}$*. Suppose also that* $2^{\aleph_0} < 2^{\aleph_1}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

We can prove a partial extension of Keisler's Theorem for more general Abstract Elementary Classes, as follows. Hypothesis (3) below corresponds to one of the cases given by Burgess's theorem for analytic equivalence relations (see [11], Theorem 9.1.5). Theorem 0.4 follows from Theorem 4.6 below.

Theorem 0.4. *Suppose that* $\mathbf{K}$ *is an Abstract Elementary Class such that*

1. *the set of reals coding countable structures in* $\mathbf{K}$ *and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$ *are both analytic;*
2. $\mathbf{K}$ *satisfies amalgamation for countable models;*

[2] This definition does not extend to uncountable A, see page 138 of [1]

[3] This requirement that M is a model is essential; Example 3.17 of [1] is ω-stable but there are countable atomic A with $|S_{at}(A)| = 2^{\aleph_0}$

3. *there is a countable model in* $\mathbf{K}$ *over which there is a perfect set of reals coding inequivalent Galois types.*

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

Hypothesis (1) of Theorem 0.4 can be relaxed if one is willing to assume the existence of a Woodin cardinal below a measurable cardinal.

Though the approach here can very likely be applied more generally, we restrict our attention in this paper to the contexts of Theorems 0.3 and 0.4.

1 Iterations

The main technical tool in this paper is the iterated generic elementary embedding induced by the nonstationary ideal on ω_1, which we will denote by NS_{ω_1}. We are using this as a device to reproduce Keisler's constructions for expanding a countable model of set theory in such a way that sets in the original model get new members in the extension if and only if they are uncountable from the point of view of the original model. Though this will not be relevant here, we note that this these iterated embeddings and their relatives play a fundamental role in Woodin's $\mathbb{P}_{\max}$ forcing [24]. Most of this section is a condensed version of Section 1 of [20].

Recall that NS_{ω_1} is closed under countable unions. Moreover, Fodor's Lemma (see, for instance, [14]) says that for any stationary $A \subseteq \omega_1$, if $f\colon A \to \omega_1$ is regressive (i.e., $f(\alpha) < \alpha$ for all $\alpha \in A$), then f is constant on a stationary set. Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ over a ZFC model M gives rise to an M-normal ultrafilter U on ω_1^M (i.e., every regressive function on ω_1^M in M is constant on a set in U). Given such M and U, we can form the generic ultrapower $\mathrm{Ult}(M,U)$, which consists of all functions in M with domain ω_1^M, where for any two such functions f, g, and any relation R in $\{=,\in\}$, fRg in $\mathrm{Ult}(M,U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha)Rg(\alpha)\} \in U$. By convention, we identify the well-founded part of the ultrapower $\mathrm{Ult}(M,U)$ with its Mostowski collapse. The corresponding elementary embedding $j\colon M \to \mathrm{Ult}(M,U)$ (where each element of M is mapped to the equivalence class of its corresponding constant function on ω_1^M) has critical point (i.e., first ordinal moved) ω_1^M (see Fact 1.2 and the discussion before). We say that such an embedding is *derived by forcing with* $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ *over* M. Fodor's Lemma implies that the identity function represents the ordinal ω_1^M in the ultrapower. It follows then by the definition of $\mathrm{Ult}(M,U)$ that for each $A \in \mathcal{P}(\omega_1)^M$, $A \in U$ if and only if $\omega_1^M \in j(A)$. Each ordinal $\gamma \in \omega_2^M$ is represented in $\mathrm{Ult}(M,U)$ by a function of the form $f(\alpha) = o.t.(g[\alpha])$, where $g\colon \omega_1 \to \gamma$ is a surjection (and *o.t.* stands for "ordertype"), so the ordinals of $\mathrm{Ult}(M,U)$ always contain an isomorphic copy of ω_2^M (which is less than or equal to $j(\omega_1^M)$, since each such f has range contained in ω_1^M) as an initial segment. We call such a function f a *canonical function* for γ. While it is possible to have well-founded ultrapowers of the form $\mathrm{Ult}(M,U)$ (at least assuming the existence of large cardinals), this does not always happen.

Since we want to deal with structures whose existence can be proved in ZFC, we define the fragment ZFC$^\circ$ to be the theory ZFC $-$ Powerset $-$ Replacement + "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the following scheme, which is a strengthening of ω_1-Replacement: every (possibly proper class) tree of height ω_1 definable from set parameters has a maximal branch (i.e., a branch with no proper extensions; in the cases we are concerned with, this just means a branch of length ω_1). The theory ZFC$^\circ$ holds in every structure of the form $H(\kappa)$ or V_κ, where κ is a regular cardinal greater than $2^{2^{\aleph_1}}$ (recall that $H(\kappa)$ is the collection of sets whose transitive closures have cardinality less than κ). For us, the importance of ZFC$^\circ$ is that it proves Fact 1.1 below, which implies that M is elementarily embedded in $\mathrm{Ult}(M, U)$ whenever M is a model of ZFC$^\circ$ and U is an M-ultrafilter on ω_1^M.[4] The proof of the fact is a direct application of the ω_1-Replacement-like scheme in ZFC$^\circ$.

1.1 Fact (ZFC$^\circ$). Let n be an integer. Suppose that ϕ is a formula with $n+1$ many free variables and $f_0, \ldots, f_{n-1}$ are functions with domain ω_1. Then there is a function g with domain ω_1 such that for all $\alpha < \omega_1$,

$$\exists x \phi(x, f_0(\alpha), \ldots, f_{n-1}(\alpha)) \Rightarrow \phi(g(\alpha), f_0(\alpha), \ldots, f_{n-1}(\alpha)).$$

We let $j[x]$ denote $\{j(y) \mid y \in x\}$. One direction of Fact 1.2 below follows from the fact that every partition in M of ω_1^M into ω many pieces must have one piece in the ultrafilter U, so, if x is countable then every function from ω_1 to x in M (i.e., every representative of a member of $j(x)$) must be constant on a set in U (so must represent a member of $j[x]$). For the other direction, note that if x is uncountable then any injection from ω_1 to x represents an element of $j(x) \setminus j[x]$ in the ultrapower $\mathrm{Ult}(V, U)$.

1.2 Fact. Suppose that M is a model of ZFC$^\circ$, and that $j \colon M \to \mathrm{Ult}(M, U)$ is an elementary embedding derived from forcing over M with $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then for all $x \in M$, $j(x) = j[x]$ if and only if x is countable in M.

If M is a countable model of ZFC$^\circ$ then there exist M-generic filters for the partial order $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Furthermore, if $j \colon M \to N$ is an ultrapower embedding of this form (where N may be ill-founded), then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is countable (recall that the ultrapower uses only functions from M), and there exist N-generic filters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^N$. We can continue choosing generic filters in this way for up to ω_1 many stages, defining a commuting family of elementary embeddings and using this family to take direct limits at limit stages.

We use the following formal definition.

1.3 Definition. Let M be a model of ZFC$^\circ$ and let γ be an ordinal less than or equal to ω_1. An *iteration* of M of length γ consists of models M_α ($\alpha \leq \gamma$), sets G_α ($\alpha < \gamma$) and a commuting family of elementary embeddings $j_{\alpha\beta} \colon M_\alpha \to M_\beta$ ($\alpha \leq \beta \leq \gamma$) such that

[4] An M-ultrafilter on ω_1 is a maximal proper filter contained in $\mathcal{P}(\omega_1)^M$; in the cases we are interested in, the filter is not an element of M.

- $M_0 = M$,
- each G_α is an M_α-generic filter for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^{M_\alpha}$,
- each $j_{\alpha\alpha}$ is the identity mapping,
- each $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by G_α,
- for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system

  $$\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\},$$

  and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

The models M_α in Definition 1.3 are called *iterates* of M. When the individual parts of an iteration are not important, we sometimes call the elementary embedding $j_{0\gamma}$ corresponding to an iteration an iteration itself. For instance, if we mention an iteration $j \colon M \to M^*$, we mean that j is the embedding $j_{0\gamma}$ corresponding to some iteration

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \gamma, \beta < \gamma\rangle$$

of M, and that M^* is the final model of this iteration.

1.4 Remark. We emphasize that for any countable model M of ZFC$^\circ$ there are $2^{\aleph_0}$ many M-generic ultrafilters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. It follows that there are $2^{\aleph_1}$ many iterations of M of length ω_1.

1.5 Remark. As noted above, the ordinals of $\mathrm{Ult}(M, U)$ always contain an isomorphic copy of ω_2^M as an initial segment, whenever M is a countable (wellfounded or illfounded) model of ZFC$^\circ$ and U is an M-normal ultrafilter. It follows from this that whenever

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \omega_1, \beta < \omega_1\rangle$$

is an iteration of M, $\omega_1^{M_{\omega_1}}$ contains a closed copy of ω_1 corresponding to the members of the set $\{\omega_1^{M_\alpha} : \alpha < \omega_1\}$. This set is called the *critical sequence* of the iteration.

Fact 1.6 below says that the final model of an iteration of length ω_1 is correct about uncountability. It is an immediate consequence of Fact 1.2 and the definition of iterations. This gives another proof of Corollary B on page 138 of [16]. Corollary A on page 137 can also be proved by considering ideals on other cardinals. The last sentence of Fact 1.6 follows from the remarks at the end of the second paragraph of this section. The second author observed that the absoluteness of the existence of a model in $\aleph_1$ of an arbitrary sentence is $L_{\omega_1,\omega}$ follows easily from Fact 1.6; it is shown in [6] that this argument can be carried out using Corollary A of [16].

1.6 Fact. Suppose that M is a model of ZFC°, and that M_{ω_1} is the final model of an iteration of M of length ω_1. Then for all $x \in M_{\omega_1}$, $M_{\omega_1} \models$ "x is uncountable" if and only if $\{y \mid M_{\omega_1} \models x \in y\}$ is uncountable. Furthermore, ω_2^M is a proper initial segment of $\omega_1^{M_{\omega_1}}$.

Fact 1.7 records the fact that one can easily make M_{ω_1} correct about stationarity for subsets of its ω_1 (again, this is due to Woodin[24]). Note that the notion of stationarity makes sense for any uncountable set (so in particular, for $\omega_1^{M_{\omega_1}}$ as below, even if it is ill-founded) : $Y \subseteq [X]^{\aleph_0}$ is stationary if and only if every for every function $F\colon X^{<\omega} \to X$ there is a nonempty element of Y closed under F.

1.7 Fact. Suppose that M is a model of ZFC°, $\{B_\xi : \xi < \omega_1\}$ is a partition of ω_1 into stationary sets and

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1\rangle \tag{1}$$

is an iteration of M of length ω_1. Suppose that for every $\alpha < \omega_1$ and every $A \in (\mathcal{P}(\omega_1) \setminus NS_{\omega_1})^{M_\alpha}$ there is a $\xi < \omega_1$ such that, for all $\beta \in \omega_1 \setminus \alpha$,

$$\beta \in B_\xi \Rightarrow j_{\alpha,\beta}(A) \in G_\beta.$$

Then for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary.

The following lemma gives a construction for building generic ultrapowers whose ω_1's are illfounded, though, as remarked above, they must be illfounded up to at least the ω_2 of the ground model. Given a function $f\colon \omega_1 \to \omega_1$, we let I_f be the normal ideal on ω_1 generated by sets of the form

$$\{\beta < \omega_1 \mid g(\beta) \geq f(\beta)\},$$

where g is a canonical function for an ordinal less than ω_2. Whenever $\gamma < \gamma' < \omega_2$, g is a canonical function for γ and γ' is a canonical function for γ', it follows that $\{\beta < \omega_1 \mid g(\beta) < g'(\beta)\}$ contains a club. It follows that for each $S \in \mathcal{P}(\omega_1)$, $S \in I_f$ if and only if $\{\beta \in S \mid f(\beta) \geq g(\beta)\}$ is nonstationary for some canonical function g for an element of ω_2. If $\langle \sigma_\beta : \beta < \omega_1\rangle$ is a $\diamondsuit$-sequence and $\pi\colon \omega_1 \to \omega_1 \times \omega_1$ is a bijection, then $\omega_1 \notin I_h$, where $h\colon \omega_1 \to \omega_1$ is the function defined by letting $h(\beta)$ be $o.t.(\pi[\beta]) + 1$ whenever $\pi[\beta]$ is a wellordering (and 0 otherwise). We note that $\diamondsuit$ is forced by the partial order which adds a subset of ω_1 by countable initial segments, and that this partial order does not add subsets of ω.

Lemma 1.8. *Suppose that M is a countable transitive model of* ZFC°*, and $f^*\colon \omega_1^M \to \omega_1^M$ is a function in M such that $\omega_1 \notin I_{f_0}$. Then there is an M-normal ultrafilter U such that the wellfounded ordinals of* $\mathrm{Ult}(M, U)$ *are exactly ω_2^M.*

Proof. Applying the usual construction of a M-normal ultrafilter, it suffices to show that if

- S is a subset of ω_1^M in M,
- $f\colon S \to \omega_1^M$,
- $S \notin I_h$,
- $\{T_\alpha : \alpha \in \omega_1^M\}$ is a collection of stationary subsets of S in ω_1 whose diagonal union is S,

then there exist $\alpha < \omega_1^M$ and $S' \subseteq T_\alpha$ and $f'\colon S' \to \omega_1$ in M such that

- for all $\beta \in S'$, $f'(\beta) < f(\beta)$,
- $S' \notin I_{f'}$.

This implication gives a recipe for building an M-normal filter with the property that every function in M from ω_1^M to the ordinals either represents an ordinal below ω_2^M or dominates on a set in the filter another function which does not represent an ordinal below ω_2^M. The recipe uses an enumeration $\{h_n : n \in \omega\}$ of $(\omega_1^{\omega_1})^M$. In each step, starting with $f = f^*$ and $S = \omega_1$, it applies the implication above to $\min\{f, h_n\}$ (for the next n, considered in order) if $S \notin I_{\min f, h_n}$, and to f otherwise.

To see that the implication holds, fix f and S as given. Since I_f is normal and $S \notin I_f$, there is an α such that $S \cap T_\alpha \notin I_f$. Let S_0 be the set of $\beta \in S \cap T_\alpha$ for which $f(\beta)$ is a successor ordinal. If S_0 is not in I_f, then let $S' = S_0$ and let $f'(\beta) = f(\beta) - 1$ for $\beta \in S'$. Then since adding 1 to the values of any canonical function for any $\gamma < \omega_2$ gives a canonical function for $\gamma + 1$, we have that $S' \notin I_{f'}$.

If $S_0 \in I_f$, there is an I_f-positive $S_1 \subseteq S \cap T_\alpha$ such that $f(\beta)$ is a limit ordinal for all $\beta \in S_1$. Let $f_n\colon S_1 \to \omega_1$ $(n \in \omega)$ be functions such that for each $\beta \in S_1$, $\langle f_n(\beta) : n < \omega\rangle$ is an increasing sequence with supremum $f(\beta)$. It suffices to see that $S_1 \notin I_{f_n}$ for some $n \in \omega$. Supposing towards a contradiction that $S_1 \in I_{f_n}$ for each $n \in \omega$, fix, for each n a canonical function g_n (for some ordinal $\gamma_n < \omega_2^M$) such that $\{\beta \in S_1 \mid f_n(\beta) \geq g_n(\beta)\}$ is nonstationary. Let γ be an element of ω_2^M greater than all the g_n's, and fix a canonical function g for γ. Then for each $n \in \omega$ the set $\{\beta \in S_1 \mid f_n(\beta) > g(\beta)\}$ is nonstationary, which means that the set $\{\beta \in S_1 \mid f(\beta) > g(\beta)\}$ is nonstationary, which means that $S_1 \in I_f$, giving a contradiction. $\square$

We conclude this section by mentioning the following consequence of large cardinals (due to Woodin, but see [7]), which we be used in Remark 4.7 and Section 5. Given an ordinal δ, $Col(\omega_1, {<}\delta)$ is the partial order which consists of countable partial functions $f\colon \omega_1 \times \delta \to \delta$, with the stipulation that $f(\alpha, \beta) < \beta$ for all $(\alpha, \beta) \in \mathrm{dom}(f)$, ordered by inclusion. This partial order preserves stationary subsets of ω_1 and does not add countable sets of ordinals. If δ is a regular cardinal, then δ is the ω_2 of any forcing extension by $Col(\omega_1, {<}\delta)$.

A model whose iterates are all wellfounded is said to be *iterable.*

Theorem 1.9. *Suppose that κ is a regular cardinal, $\lambda < \kappa$ is a measurable cardinal and $\delta < \lambda$ is a Woodin cardinal. Let X be a countable elementary submodel of either V_κ or $H(\kappa)$, with δ and λ in X. Let M be the transitive collapse of X, and let $\bar{\delta}$ be the image of δ under this collapse. Let $g \subseteq Col(\omega_1, {<}\bar{\delta})$ be an M-generic filter. Then $M[g]$ is iterable.*

2 $L_{\omega_1,\omega}(\mathrm{aa})$

Briefly, the logic $L_{\omega_1,\omega}$ is the extension of first order logic where one allows countable conjunctions and disjunctions of formulas, except in the case where this would give rise to formulas with infinitely many free variables. Each formula in $L_{\omega_1,\omega}$ has a *rank*, the number (less than ω_1) of steps it takes to construct the formula from atomic formulas (see the appendix to [2]). More explicitly, we may think of sentences of $L_{\omega_1,\omega}$ as well-founded trees of height of at most ω; then the rank of a sentence is just the rank of the corresponding tree in the sense of Section 3. An ill-founded model of ZFC$^\circ$ can contain objects which it thinks are sentences of $L_{\omega_1,\omega}$ which are really not, i.e., if the rank of the sentence as computed in the model is an ill-founded ordinal of the model. On the other hand, if a (real) sentence ϕ of $L_{\omega_1,\omega}$ exists in an ω-model M of ZFC$^\circ$, then M computes the rank correctly, and is therefore well-founded at least up the rank of ϕ. Furthermore, M correctly verifies whether the models that it sees satisfy ϕ. In both cases, the computation of the rank and the verification of the truth value, M runs exactly the same process that is carried out in V.

The logic $L_{\omega_1,\omega}(\mathrm{aa})$ extends $L_{\omega_1,\omega}$ by adding the quantifier aa, where $\mathrm{aa}x \in [X]^{\aleph_0}\,\phi$ means "for stationarily many countable $x \subseteq X$, ϕ holds", i.e., for any function $f\colon X^{<\omega} \to X$, there is a countable $x \subseteq X$ closed under f such that x satisfies ϕ. Note that "there exist uncountably many $x \in X$ such that ϕ holds" can be expressed using aa. Note also that if M is a model of ZFC$^\circ$ as in conclusion of Fact 1.7, i.e., such that for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary, then if X is a set in M of cardinality $\aleph_1$ (in M) and Y is a subset of $[X]^{\aleph_0}$ in M, then $M_{\omega_1} \models$ "Y is stationary" if and only if Y is stationary

The second parts of the equivalences in the following theorems are $\underset{\sim}{\Sigma}^1_1$, and therefore absolute. The forward directions simply involve taking the transitive collapse of a countable elementary submodel of suitable initial segment of the universe. The reverse directions involve building iterations as in the previous section (using Fact 1.7 for correctness about stationarity). Since the final models of these iterations are well-founded up to at least the ω_2 of the corresponding original models, they verify correctly truth for members of the set F for the models that they see.

Theorem 2.1. *Given a sentence ϕ of $L_{\omega_1,\omega}(\mathrm{aa})$, the existence of a model of size $\aleph_1$ with a member satisfying ϕ is equivalent to the existence of a countable model of* ZFC$^\circ$ *containing $\{\phi\} \cup \omega$ which thinks there is a model of size $\aleph_1$ with a member satisfying ϕ.*

Theorem 2.2. *Given a countable fragment F of $L_{\omega_1,\omega}(\text{aa})$, the existence of a model of size $\aleph_1$ satisfying $\aleph_1$-many F-types is equivalent to the existence of a countable model of* ZFC$^\circ$ *containing $F \cup \{F\} \cup \omega$ which thinks there is a model of size $\aleph_1$ satisfying $\aleph_1$-many F-types.*

We prove in Theorem 2.4 below that the second part of the equivalence in the previous theorem implies that there are $2^{\aleph_1}$ many models of size $\aleph_1$, pairwise satisfying only countably many F-types in common. First we note an easier argument for getting $\aleph_1$ many such models.

Suppose that M is an ω-model of ZFC$^\circ$ and $\bar{x} = \langle x_\alpha : \alpha < \omega_1^M \rangle$ is a sequence of distinct subsets of ω in M. Then given any iteration of M as above, $\bar{x}$ will be an initial segment of $j_{0,\omega_1}(\bar{x}) = \langle x_\alpha : \alpha < \omega_1^{M_{\omega_1}} \rangle$, and $x_\alpha \notin M_\beta$ whenever $\alpha \geq \omega_1^{M_\beta}$ (by the remarks before Fact 1.2).

Furthermore, if A is any countable set of reals not in M, one can easily build an iteration of M such that $A \cap M_{\omega_1} = \emptyset$. Now let F be a countable fragment of $L_{\omega_1,\omega}(\text{aa})$, and let M be a ω-model of ZFC$^\circ$ in which F is countable, which thinks there exists a model N of size $\aleph_1$ realizing uncountably many F-types. Then there are uncountably many iterations $\{j^\xi : \xi < \omega_1\}$ of M producing models $\{M^\xi_{\omega_1} : \xi < \omega_1\}$ such that the models $M^\xi_{\omega_1}$ pairwise have only the reals from M in common, and thus the models $j^\xi(N)$ pairwise realize just countably many F-types in common.

To get $2^{\aleph_1}$ many uncountable iterates pairwise having just countably many reals in common, we use Theorem 2.3 below. Note that one can force MA$_{\aleph_1}$ (the restriction of Martin's Axiom which asserts the existence of a filter meeting any $\aleph_1$ many maximal antichains from a c.c.c. partial order) to hold over any countable model of ZFC$^\circ$. By "distinct iterations" we mean literally iterations that are not the same set, formally speaking. In particular, this means (using the notation from Theorem 2.3) that there is some β such that $G_\beta \neq G'_\beta$. When β is minimal with this property, $M_\beta = M'_\beta$ and there is a set $A \in \mathcal{P}(\omega_1)^{M_\beta}$ such that $A \in G_\beta$ and $\omega_1^{M_\beta} \setminus A \in G'_\beta$, since G_β and G'_β are distinct M_β-ultrafilters.

Theorem 2.3 (Larson [19])**.** *If M is a countable model of* ZFC$^\circ$ + MA$_{\aleph_1}$ *and*

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle$$

and

$$\langle M'_\alpha, G'_\beta, j'_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle$$

are two distinct iterations of M, then

$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} \subset M_\beta,$$

where β is least such that $G_\beta \neq G'_\beta$.

For the reader's convenience, we sketch the proof of the version of Theorem 2.3 for iterations of length 1 (which appears in [9]). Suppose that M is a countable model of ZFC$^\circ$ + MA$_{\aleph_1}$ and let G and G' be two distinct M-generic filters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then there exist disjoint sets A, A' in $(\mathcal{P}(\omega_1 \setminus NS_{\omega_1})^M$ such that $A \in G$ and $A' \in G'$. Let $N = \mathrm{Ult}(M, G)$ and $N' = \mathrm{Ult}(M, G')$, and fix $x \in \mathcal{P}(\omega)^N \setminus M$ and $x' \in \mathcal{P}(\omega)^{N'} \setminus M$. Then there exist functions $f\colon A \to \mathcal{P}(\omega)^M$ and $f'\colon A' \to \mathcal{P}(\omega)^M$ representing x in N and x' in N' respectively. Applying Fodor's Lemma we see that, since x and x' are not in M, there exist $B \subseteq A$ and $B' \subseteq A'$ in G and G' respectively on which f and f' (respectively) are injective. Applying Fodor's Lemma again we can thin B and B' to sets C and C' on which the ranges of f and f' are disjoint and contain only infinite, co-infinite sets, by subtracting nonstationary sets. Finally, it is a consequence of $\mathrm{MA}_{\aleph_1}$ (see [15], for instance) that for any two disjoint sets of infinite, co-infinite subsets of ω, there is a subset of ω which intersects each member of the first set infinitely, and no member of the second set infinitely. Thus if M satisfies $\mathrm{MA}_{\aleph_1}$ there is such a $z \subseteq \omega$ in M with respect to the ranges of $f\restriction C$ and $f\restriction C'$, which means that $x \cap z$ is infinite and $x' \cap z$ is not.

Using this, one gets the following version Keisler's theorem (see Fact 18.15 of [1]), for $L_{\omega_1,\omega}(\mathrm{aa})$.

Theorem 2.4. *Let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$. If there exists a model of cardinality $\aleph_1$ realizing uncountably many F-types, there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many F-types in common.*

Proof. Let N be a model of cardinality $\aleph_1$ realizing uncountably many F-types, let X be a countable elementary submodel of $H((2^{2^{\aleph_1}})^+)$ containing $\{N\}$ and the transitive closure of $\{F\}$. Let M be the transitive collapse of X, and let N_0 be the image of N under this collapse. Let M' be a c.c.c. forcing extension of M satisfying Martin's Axiom. By choosing a pair of distinct generic ultrafilters for each model we can build a tree of iterates of M' giving rise to $2^{\aleph_1}$ many distinct iterations of M' of length ω_1 (as in Remark 1.4). Since F-types can be coded by reals using an enumeration of F in M, the images of N_0 under these iterations will pairwise realize just countably many F-types in common, by Theorem 2.3. $\square$

If one assumes in addition that $2^{\aleph_0} < 2^{\aleph_1}$, then, as in Theorem 18.16 of [1], one gets that if there exists a model of cardinality $\aleph_1$ realizing uncountably many types over some countable subset, then there exists a $2^{\aleph_1}$-sized family of nonisomorphic models. That is, if there is an uncountable model N with a countable subset A over which uncountably many types are realized, then there are models N_f ($f \in 2^{\aleph_1}$) all containing the same countable set A and all realizing different sets of types over A, so that any isomorphisms of any two N_{f_1} and N_{f_2} into a third N_{f_3} must map A to different sets (which is impossible if $2^{\aleph_1} > 2^{\aleph_0}$).

We conclude this section by showing that a strengthening of Lemma 5.1.8 of [1] can be proved using Lemma 1.8.

Lemma 2.5. *Suppose that ϕ is a sentence of $L_{\omega_1,\omega}(\mathrm{aa})$ in a language with a binary predicate $<$, and suppose that there is a model M of ϕ for which the* *ordertype of* $(M,<)$ *is* ω_1*. Then there is a model* M' *of* ϕ *of cardinality* $\aleph_1$ *such that* $(M',<)$ *imbeds* $\mathbb{Q}$*. Furthermore, if* θ *is a regular cardinal greater than* $2^{2^{\aleph_1}}$*, then* M' *can be taken the be an element of a model* N *of* ZFC$^\circ$ *such that* M' *satisfies every* Σ_1 *formula in* N *that* M *does in* $H(\theta)$*.*

Proof. Let X be a countable elementary submodel of $H(\theta)$ with $M \in X$. Let N_0 be the transitive collapse of X and let M_0 be the image of M under this collapse. Let N_1 be a forcing extension of N_0 (with the same reals) satisfying $\diamondsuit$. Then M_0 satisfies every Σ_1 formula in N_1 that M does in $H(\theta)$. Applying Lemma 1.8 (for the first step of the iteration) and Fact 1.7 (for the rest), we can find an iterate N of N_1 of cardinality $\aleph_1$ whose wellfounded ordinals are exactly $\omega_2^{N_1}$. Letting M' be the image of M_0 under this iteration, we have that $(M',<)$ is isomorphic to ω_1^N, which embeds $\mathbb{Q}$ as it is illfounded. □

3 Analytic equivalence relations

For our purposes, a *tree* is a set of finite sequences closed under initial segments. If $T \subseteq X^{<\omega}$ is a tree, for some set X, then $[T]$ is the set of $x \in X^\omega$ such that $x{\restriction}n \in T$ for all $n \in \omega$. If $T \subseteq (X \times Y)^{<\omega}$, for some sets X and Y, then the *projection* of T, $p[T]$ is the set of $f \in X^\omega$ such that for some $g \in Y^\omega$, $(f,g) \in [T]$ (this definition involves a standard identification of pairs of sequences with sequences of pairs). For any positive $n \in \omega$, a subset of $(\omega^\omega)^n$ is *analytic* if it has the form $p[T]$ for some tree $T \subseteq (\omega^n \times \omega)^{<\omega}$.

Recall that for a tree $T \subseteq X^{<\omega}$ for some set X, the ranking function $rank_T\colon T \to Ord \cup \{\infty\}$ is defined in such a way that for all $t \in T$, $rank_T(t)$ is the smallest ordinal α such that $\alpha > rank_T(s)$ for all proper extensions s of t in T, and $rank_T(t) = \infty$ if no such α exists (which happens if and only if $rank_T(s) = \infty$ for some proper extension s of t). We write $rank(T)$ for $rank_T(\langle\rangle)$. Then $rank(T) = \infty$ if and only if T has an infinite branch.

Now suppose that M is an ω-model of ZFC$^\circ$, and $T \subseteq X^{<\omega}$ is a tree in M, for some X in M. If $rank(T)^M = \infty$, then there is an infinite branch through T in M. If $rank(T)^M$ is in the well-founded part of M, then there is no infinite branch through T (in V). It follows easily from the definition of $rank(T)$ that if $rank(T)^M$ is an ill-founded ordinal of M, then T has an infinite branch in V but no infinite branch in M. If M is ill-founded then there will be trees T in M for which this happens.

Given sets X, Y, a tree $T \subseteq (X \times Y)^{<\omega}$ and $s^* \in X^{<\omega}$, T_{s^*} is the set of $(s,t) \in T$ such that s is compatible with s^* (i.e., one of them extends the other).

Lemma 3.1. *Suppose that* M *is a (possibly ill-founded)* ω*-model of* ZFC$^\circ$*, and that* $T \subseteq (X \times Y)^{<\omega}$ *is a tree in* M*, for some sets* X *and* Y*. Suppose that* x *is the unique element of* $p[T]$*. Then* $x \in M$*.*

Proof. Since $p[T]$ is nonempty, $rank(T)^M$ cannot be in the well-founded part of M. If $rank(T)^M = \infty$, then $[T] \cap M$ is nonempty, which means that $p[T] \cap M$ is nonempty. Suppose then that $rank(T)^M$ is an ill-founded ordinal of M. Then, starting with with $\langle\rangle$, M can find all the initial segments of x by the following process. Suppose that $s \in X^{<\omega}$ is an initial segment of x. Then $rank(T_s)^M$ is an ill-founded ordinal of M. Since s is an initial segment of the unique element of $p[T]$, the unique integer n such that $s^\frown\langle n\rangle$ is an initial segment of x is also the unique integer n such that

$$\sup\{rank_T^M(s^\frown\langle n\rangle, t) : (s^\frown\langle n\rangle, t) \in T\}$$

is greater than

$$\sup\{rank_T^M(s^\frown\langle m\rangle, t) : (s^\frown\langle m\rangle, t) \in T\}$$

for all $m \in \omega \setminus \{n\}$, since the former set contains ill-founded ordinals of M and the latter contains only well-founded ordinals. $\square$

An interesting aspect of the proof just given is that it does not give an element of $[T]$ in M.

Now suppose that E is an analytic equivalence relation on an analytic set $X \subseteq \omega^\omega$. By the Burgess Trichotomy Theorem (Theorem 9.1.5 of [11]), either E has at most $\aleph_1$ many equivalence classes, or there is a perfect set P consisting of E-inequivalent members of X. The following lemma shows that in this second case, if M is an ω-model containing codes for E and P, and $x \in \omega^\omega \cap M$ is E-equivalent to a member of P, then this member of P is also in M. The lemma follows from Lemma 3.1 plus the fact that the set of members of P which are E-equivalent to x is an analytic set with a unique member.

Lemma 3.2. *Suppose that M is a (possibly ill-founded) ω-model of* ZFC°*, and E is an analytic equivalence relation on ω^ω which is the projection of a tree T on $\omega \times \omega \times \omega$ in M. Suppose that P is a perfect set of E-inequivalent members of ω^ω such that $P = [S]$ for a tree $S \subseteq \omega^{<\omega}$ in M. Let $x \in M \cap \omega^\omega$ be such that xEy for some $y \in P$. Then $y \in M$.*

4 Abstract Elementary Classes

In this section we work with an abstract elementary class $\mathbf{K}$ in a countable vocabulary τ with Löwenheim number $\aleph_0$. To study the countable members of $\mathbf{K}$ in descriptive set theoretic structure we regard them as collections of relations (indexed by τ) on ω. The class of countable structures is a Polish spaces and classes defined by a sentence $L_{\omega_1,\omega}$ are Borel.

We study classes satisfying the following additional condition.

Definition 4.1. *An abstract elementary class* $\mathbf{K}$ *is* analytically presented *if the set of countable models in* $\mathbf{K}$*, and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$*, are both analytic.*

This requirement is not as *ad hoc* as it might seem. Shelah's presentation theorem (Theorem 4.15 of [1]) asserts that any AEC of τ-structures with countable Löwenheim-Skolem can be presented as the reducts to τ of models of a first order theory in a countable language τ' which omit a family of at most $2^{\aleph_0}$-types. If the collection of omitted types is countable, we call these $PC\Gamma(\aleph_0, \aleph_0)$ classes[5]. Sentences in $L_{\omega_1,\omega}(Q)$ are $PC\Gamma(\aleph_0, \aleph_0)$. To say an AEC has a $PC\Gamma(\aleph_0, \aleph_0)$ implies as well that the strong substructure relation is $PC\Gamma(\aleph_0, \aleph_0)$.

Note first than any $PC\Gamma(\aleph_0, \aleph_0)$-presented AEC is analytically presented, as omission of a countable family of types in τ' is Borel, and taking the reduct to τ makes the class of countable models analytic.

The assumption that $\mathbf{K}$ is analytically presented implies that the countable models of $\mathbf{K}$ are the countable models of a $PC\Gamma(\aleph_0, \aleph_0)$ class. Since any AEC $\mathbf{K}$ is determined by its restriction to models of cardinality at most the Löwenheim number, this restriction determines $\mathbf{K}$. But it does not follow immediately that an analytically presented $\mathbf{K}$ is a $PC\Gamma(\aleph_0, \aleph_0)$ class. Example 4.29 of [1] is a suggestive but not determinative example. Models in $\aleph_1$ of $PC\Gamma(\aleph_0, \aleph_0)$-classes are studied extensively in the first chapter of [23].

Following [1] we define for $\mathbf{K}$ a reflexive and symmetric relation $\sim_0$ on the set of triples of the form (M, a, N), where M and N are countable structures in $\mathbf{K}$ with $M \prec_{\mathbf{K}} N$, and $a \in N \setminus M$. We say that $(M_0, a_0, N_0) \sim_0 (M_1, a_1, N_1)$ if $M_0 = M_1$ and there exist a structure $N \in \mathbf{K}$ and strong embeddings $f_0 \colon N_0 \to N$ and $f_1 \colon N_1 \to N$ such that $f_0 \restriction M_0 = f_1 \restriction M_1$ and $f_0(a_0) = f_1(a_1)$. We let $\sim$ be the transitive closure of $\sim_0$. The equivalence classes of $\sim$ are called *Galois types*. If an abstract elementary class is given syntactically the Galois types over a countable M refine the syntactic types and in general there may be more Galois types than syntactic types (e.g. [3]).

There is a natural coding of triples (M, a, N) as above by elements of ω^ω, and for analytically presented AEC the set B consisting of those $x \in \omega^\omega$ coding such a triple is an analytic set. Furthermore, if we let E be the equivalence relation on B such that xEy if and only if the triples coded by x and y are $\sim$-equivalent, then E is analytic. For a given model M, we let E_M be the equivalence relation E restricted to the set B_M consisting of codes for triples whose first element is M. Then E_M is also analytic.

By Burgess's Trichotomy, for each such M there are either at most $\aleph_1$ many E_M-equivalence classes, or a perfect set of E_M-inequivalent reals. For the syntactic types discussed in the earlier sections the intermediate possibility of $\aleph_1$-types is impossible, as for each countable fragment of ($L_{\omega_1,\omega}$, $L_{\omega_1,\omega}(Q)$, $L_{\omega_1,\omega}(aa)$) the set of types is Borel (See 4.4.13 in [21].) But for analytically presented AEC all three parts of the trichotomy occur (see Example 4.4 below) and Theorem 0.2 does not generalize in full. Following [23], we use the following definitions.

4.2 Definition. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be *Galois ω-stable* if for each countable $M \in \mathbf{K}$, E_M has countably many equivalence classes, and *almost Galois ω-stable*[6] if for each countable $M \in \mathbf{K}$, E_M has at most $\aleph_1$ many equivalence classes.

[5] Shelah writes $PC_{\aleph_0}$ or $PC(\aleph_0, \aleph_0)$, suppressing the type omission, and Keisler writes PC_δ over $L_{\omega_1,\omega}$ for this notion.

[6] Shelah, e.g. [13] calls this notion 'weak stability'. Since we discuss a second notion of weak stability ([12] where the 'weak' is more intrinsic, we change to 'almost' here.

In the infinitary case and certainly for AEC, we must consider stability with respect to the cardinalities of the models under consideration; the analog for Galois types of the first order theorem that ω-stability implies stability in all powers fails except under very restrictive conditions. Baldwin and Kolesnikov [3] exhibit complete sentences that are ω-Galois stable but not Galois stable in $\aleph_1$.

4.3 Example. Consider the abstract elementary class $(\mathbf{K}, \prec)$ where $\mathbf{K}$ is the class of well-order types of length $\leq \omega_1$ and $\prec$ is initial segment. $(\mathbf{K}, \prec)$ has amalgamation and joint embedding in $\aleph_0$, is almost Galois ω-stable and $\aleph_1$-categorical.

In view of Example 4.3, there is no hope of a direct generalization of Theorem 0.2 to arbitrary Abstract Elementary Classes. The existence of almost Galois ω-stable but not Galois ω-stable classes is one obstruction. This example seems extreme as there are no models beyond $\aleph_1$ and no nice syntactic description of the class. In particular it is not analytically presented. But, we can find apparently more tractable examples of weak ω-Galois stability.

A linear order L is 1-transitive if for any a, b in L, there is an automorphism of L taking a to b. The class of groupable (equivalently 1-transitive) linear orders has exactly $\aleph_1$ countable models. (See Corollary 8.6 of [22].) The following example is a variant by Jarden of a somewhat less natural version in Chapter 1 of [23].

4.4 Example. Let $(\mathbf{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order with $M \prec N$ if $M \subseteq N$ and no component is extended. Since there are only $\aleph_1$-isomorphism types of components this class is almost Galois ω-stable. This AEC is analytically presented and definable as a reduct of a class in $L(Q)$. But it has $2^{\aleph_1}$ models in $\aleph_1$ and $2^{\aleph_0}$ models in $\aleph_0$.

This example is unsatisfactory in two respects : it uses the Q quantifier, and it involves taking a reduct.

We sketch an argument (told to us by Kesälä) that every almost ω-Galois stable complete sentence of $L_{\omega_1,\omega}$ is ω-Galois stable. Hyttinen and Kesälä introduced the important notions: finite character and weak Galois type. An AEC $\mathbf{K}$ has *finite character* if for $M \subseteq N$ with $M, N \in \mathbf{K}$: if for every finite $\boldsymbol{a} \in M$ there is a $\mathbf{K}$-embedding of M into N fixing $\boldsymbol{a}$, then $M \prec_{\mathbf{K}} N$. The key point is that any sentence of $L_{\omega_1,\omega}$ has finite and any such AEC is very close to $L_{\omega_1,\omega}$. In particular, generally speaking sentences of $L_{\omega_1,\omega}(Q)$ do not have finite character. Two points have the same *weak Galois type* over a model M if they have the same Galois type over every finite subset of M.

It follows easily from work of Kueker [18] and Hyttinen-Kesälä [12] that *for countable models* of an AEC with finite character satisfying the amalgamation and joint embedding properties , almost Galois ω-stability implies Galois ω-stability. Here is the argument. Hyttinen and Kesala call such an AEC and *weakly Galois ω-stable* if there are only countably many weak types over each countable model. For such classes, Hyttinen and Kesala show, if two elements have the same weak Galois type over a *countable* model M they have the same Galois type over M. Kueker proves that for *complete* sentences of $L_{\omega_1,\omega}$ satisfying these conditions, points a and b have the same weak-Galois type over a countable model M if and only if they have the same $L_{\infty,\omega}$ type over M. Thus for countable models of such sentences, syntactic ω-stability implies Galois ω-stability. Since we noted above that almost Galois ω-stability implies syntactic ω-stability (the set of syntactic types is Borel and so can't have cardinality $\aleph_1$), we get the following.

4.5 Fact. If a complete sentence in $L_{\omega_1,\omega}$-sentence, satisfying amalgamation and joint embedding, is almost Galois ω-stable then it is Galois ω-stable.

Baldwin, Larson, and Shelah [4] (see Theorem 5.4 below) have shown a related fact: if a $PC\Gamma(\aleph_0, \aleph_0)$ class satisfying amalgamation *has only countably many models in* $\aleph_1$ then almost Galois ω-stable implies Galois ω-stable.

We deal with the case that there is a perfect set of E_M-inequivalent reals, for some M. This perfect set plays roughly the role that the uncountable set of types played in Theorem 2.4. Since a Galois type is not a real but a set of reals, we cannot reproduce the same argument from an uncountable set of Galois types, but rather need a perfect set.

In the following theorem, we do not assume that $\mathbf{K}$ satisfies amalgamation or the joint embedding property. However, one would typically use amalgamation to obtain hypothesis (4) of the theorem.

Theorem 4.6. *Suppose that*

1. $\mathbf{K}$ *is an analytically presentable abstract elementary class;*
2. N *is a* $\mathbf{K}$*-structure of cardinality* $\aleph_1$*, and* N_0 *is a countable structure with* $N_0 \prec_{\mathbf{K}} N$*;*
3. P *is a perfect set of* E_{N_0}*-inequivalent members of* ω^ω*;*
4. *uncountably many members of* P *code triples which are* $\sim$*-equivalent to triples* (N_0, a, N') *such that* $N' \prec_{\mathbf{K}} N$*.*

Then there exists a family of $2^{\aleph_1}$ *many* $\mathbf{K}$*-structures of cardinality* $\aleph_1$*, each containing* N_0 *and pairwise realizing just countably many* P*-classes in common.*

Proof. Fix a regular $\kappa > 2^{2^{\aleph_1}}$, and let Y be a countable elementary submodel of $H(\kappa)$ with $\mathbf{K} \cap H(\aleph_1)$, N_0, N and P in Y. Let M^* be the transitive collapse of Y, and let N^* be the image of N under this transitive collapse. There is a tree $S \subseteq \omega^{<\omega}$ in M^* such that $P = [S]$. Let M_0 be a forcing extension of M^* satisfying $\mathrm{MA}_{\aleph_1}$. Let X be the set of reals of $M_0 \cap P$ coding triples which are $\sim$-equivalent to triples (N_0, a, N') with $N' \prec_{\mathbf{K}} N^*$. Then X is uncountable in M_0. By Theorem 2.3, there are $2^{\aleph_1}$ many iterates of M_0 pairwise having just countably many reals in common. Suppose that M^1 and M^2 are two such iterates via embeddings j_1 and j_2, and that N^1 and N^2 are the corresponding images of N^*. Then there is a real $y \in j_1(X)$ such that $y \notin N^2$. Since $y \in [j_1(S)] = [S]$, $y \in P$ and y codes a triple (M, a, N'), where $N' \prec_{\mathbf{K}} N^1$. If N^1 and N^2 were isomorphic via a map fixing M, then there would be a real x in N^2 which is E_M-equivalent to y. Then Lemma 3.2 implies that $y \in N^1$, which gives a contradiction. $\square$

4.7 Remark. The assumption in the previous theorem that the set of reals coding countable structures in $\mathbf{K}$ be analytic can be can be relaxed to the requirement this set of codes be universally Baire (see [10]), if one is willing to assume the existence of a Woodin cardinal with a measurable cardinal above (see [7, 8]).

5 Absoluteness of $\aleph_1$-categoricity

In first order logic, the Baldwin-Lachlan equivalence between '$\aleph_1$-categorical' and 'ω-stable with no two-cardinal models' makes the notion of $\aleph_1$-categoricity Π^1_1 and hence absolute. Shelah provided an example of an AEC, definable in $L(Q)$, which is $\aleph_1$-categorical under MA and has $2^{\aleph_1}$ models in $\aleph_1$ under $2^{\aleph_0} < 2^{\aleph_1}$. It is an open question whether there is such a non-absolute example in $L_{\omega_1,\omega}$. See [2] for some partial results.

Shelah's $L(Q)$-example fails amalgamation in $\aleph_0$ and is not ω-stable under any of the definitions. We focus here on the question of whether assuming amalgamation is enough to make $\aleph_1$-categoricity absolute for analytically presented AEC. Note that amalgamation for countable models in an analytically presented AEC and existence of an uncountable model are Σ^1_2 and so absolute In the first case, write the property, in the second write every countable model has a strong extension. This claim should not extend to AEC which are not analytically presented: if membership in $\mathbf{K}$ were Π^1_2, amalgamation would not automatically satisfy Shoenfield absoluteness.

Let us consider for a moment the case where $\mathbf{K}$ is weakly Galois ω-stable and satisfies amalgamation and the joint embedding property. In this case, there is a model in $\mathbf{K}$ of size $\aleph_1$ which realizes every Galois type over every one of its countable substructures (i.e., it is $\aleph_1$-Galois saturated). Furthermore, all such models are isomorphic. The question of $\aleph_1$-categoricity for $\mathbf{K}$ then just depends on whether $\mathbf{K}$ has a model of size $\aleph_1$ omitting some Galois type over some countable substructure. Using Theorem 1.9, one can show that this fact is also absolute to set forcing extensions, if there exist a proper class of Woodin cardinals. The argument is analogous to the proof of Theorem 2.1, except that here we need our iterates to be well-founded, so that they correctly assert that a given Galois type is not realized; getting well-foundedness for iterates of a transitive collapse of a countable elementary submodel is where Woodin cardinals are needed, and there is yet another step here, as one needs to do a preliminary forcing with the image of $Col(\omega_1, {<}\delta)$ under the transitive collapse. The Woodin cardinals are not necessary in all cases for this argument. If some model realizes just countably many Galois types over some M, and not all, since then this can be verified in an absolute way without the large cardinals, since the iterates can contain codes for the countably many types that are realized. To summarize, we have the following facts.

The second part of the following statement is $\underset{\sim}{\Sigma}^1_2$ and thus absolute.

5.1 Fact. Suppose that $\mathbf{K}$ is an analytically presented AEC. Then the following statements are equivalent.

1. There exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality $\aleph_1$ such that
   - $M \prec_{\mathbf{K}} N$;
   - the set of Galois types over M realized in N is countable;
   - some Galois type over M is not realized in N.
2. There is a countable model of ZFC° whose ω_1 is well-founded and which contains trees on ω giving rise to $\mathbf{K}$, $\prec_{\mathbf{K}}$ and the associated relation $\sim_0$, and satisfies statement (1).

The third part of the following statement is $\underset{\sim}{\Sigma}^1_2$ and thus absolute.

5.2 Fact. Suppose that $\mathbf{K}$ is an analytically presented AEC, and suppose that δ is a Woodin cardinal below a measurable cardinal. Then the following statements are equivalent.

1. There exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality $\aleph_1$ such that
   - $M \prec_{\mathbf{K}} N$;
   - some Galois type over M is not realized in N.
2. In some forcing extension via a partial order in V_δ, there exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality $\aleph_1$ such that
   - $M \prec_{\mathbf{K}} N$;
   - some Galois type over M is not realized in N.
3. There is a well-founded, countable, iterable model of ZFC° which contains trees on ω giving rise to $\mathbf{K}$, $\prec_{\mathbf{K}}$ and the associated relation $\sim_0$, and satisfies statement (1).

Now suppose that $\mathbf{K}$ is an analytically presented AEC. By the Burgess Trichotomy, we have that either $\mathbf{K}$ is Galois ω-stable or almost Galois ω-stable, or there exists a countable $M \in \mathbf{K}$ for which E_M has a perfect set of inequivalent reals. The third of these cases is $\underset{\sim}{\Sigma}^1_2$, and thus absolute. On its surface, the first is $\underset{\sim}{\Pi}^1_4$, as it says that for every M, if $M \in \mathbf{K}$ then there are countably many reals such that every suitable real is E_M-equivalent to one of them. Statements of this type are also forcing-absolute in the presence of suitable large cardinals, though not in ZFC.

Facts 5.1 and 5.2 give us the following. The word "absolutely" appears in the first part since for all we know $\mathbf{K}$ can be sometimes Galois ω-stable and sometimes only almost Galois ω-stable, and we want to rule this case out. In the second part, "absolutely" rules out the third case of the Burgess Trichotomy, but allows the case ruled out in the first part. This condition is necessary since under CH every $\mathbf{K}$ is almost Galois ω-stable.

Theorem 5.3. *Let* $\mathbf{K}$ *be an analytically presented AEC satisfying amalgamation and the joint embedding property, and having an uncountable model.*

- *If* $\mathbf{K}$ *is absolutely Galois* ω*-stable, then the* $\aleph_1$*-categoricity of* $\mathbf{K}$ *is absolute.*
- *If* $\mathbf{K}$ *is absolutely almost Galois* ω*-stable, and* δ *is a Woodin cardinal below a measurable cardinal, then the* $\aleph_1$*-categoricity of* $\mathbf{K}$ *is absolute to forcing extensions by partial orders in* V_δ*.*

As mentioned above, the second part of the previous theorem has recently been improved (see [4]).

Theorem 5.4. *Suppose that* $\mathbf{K}$ *is an analytically presented AEC satisfying amalgamation and the joint embedding property, and having an uncountable model. Suppose further that* $\mathbf{K}$ *is not Galois* ω*-stable, and that for no countable* $M \in \mathbf{K}$ *are there perfectly many Galois types over* M*. Then* $\mathbf{K}$ *has at least* $\aleph_1$ *many nonisomorphic models of cardinality* $\aleph_1$*.*

5.5 Remark. We should point out that our absoluteness results in this section and the previous one relied only on the fact that the Galois types are induced by an analytic equivalence relation. In the same way, the results of Section 2 were analyzing Borel equivalence relations. Each approach then can be applied much more generally, though we have no applications for this degree of generality at this time.

6 Questions

The following questions have been left unresolved.

6.1 Question. Is there an example like Example 4.3 (Example 4.14 of [1]) (i.e., $\aleph_1$-categorical, satisfying amalgamation and joint embedding but not ω-Galois stable) where the set of codes for countable models is analytic?

6.2 Question. Can there be an absolutely weakly Galois ω-stable analytically presented AEC whose Galois ω-stability (or lack thereof) is not absolute? Comsider Fact 4.5 and the succeeding paragraph.

6.3 Question. Is there a $PC\Gamma(\aleph_0, \aleph_0)$ AEC, which is almost Galois ω-stable, not Galois ω-stable and with κ models in $\aleph_1$, $\aleph_1 \leq \kappa < 2^{\aleph_1}$?

The answer to the following question is certainly negative, though we do not yet have a proof.

6.4 Question. Suppose that $\mathbf{K}$ is an analytically presented AEC which is Galois ω-stable. Does it follow then that the countable models of $\mathbf{K}$ are the reducts of the countable models of an ω-stable $PC\Gamma(\aleph_0, \aleph_0)$ AEC?

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