Small-sum pairs in abelian groups

Reza Akhtar
Dept. of Mathematics and Statistics
Miami University, Oxford, OH 45056, USA
reza@calico.mth.muohio.edu

Paul Larson
Dept. of Mathematics and Statistics
Miami University, Oxford, OH 45056, USA
larsonpb@muohio.edu

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Abstract

Let G be an abelian group and A, B two subsets of equal size k such that $A+A$ and $A+B$ both have size $2k-1$. Answering a question of Bihani and Jin, we prove that if $A+B$ is aperiodic, then A is a translate of B. We also show that if $A+B$ is periodic, this conclusion need not hold, and give an explicit description of the various counterexamples.

1 Introduction

Let G be an abelian group, written additively. If A and B are subsets of G, we write $A+B$ to mean $\{a+b : a \in A,\ b \in B\}$, and similarly for $A-B$. We use $A \setminus B$ to denote set difference. In their study [1] of sets of natural numbers with small upper Banach density, Bihani and Jin asked the following in the context of cyclic groups:

Question 1. Let G be an abelian group. Given a pair (X, Y) of subsets of G such that $|X| = |Y| = k$ and $|X+X| = |X+Y| = 2k-1$, are X and Y always a translates of each other; that is, does there exist $h \in G$ such that $Y = X + h$?

It is easily seen that this question cannot always be answered in the affirmative. To see this, let G be any abelian group of odd order $2k-1 \geq 5$ and A, B any two subsets of G of order k. Since $|A| + |B| > |G|$, a rudimentary combinatorial result (cf. [7, Lemma 2.1] or Lemma 2.7 below) implies that $A + B = G$, regardless of whether A and B are not translates of each other. It is far less obvious that all counterexamples are, in a suitable sense, based on this example.

The goal of this paper is to answer Question 1 in the affirmative when $X + Y$ is aperiodic and to provide explicit descriptions of counterexamples when $X + Y$ is periodic. We note that all our work is valid for arbitrary abelian groups G.

The key tool we use is the theory of *critical pairs*. A critical pair in an abelian group H is a pair (A, B) of subsets such that $|A+B| = |A|+|B|-1$; an essentially complete description of all such pairs was given by Vosper [6] for groups of prime order and by Kemperman [5] (see also [2], [3], [4]) for general abelian groups. We show how to use Kemperman's results when $X + Y$ is aperiodic and Kneser's Theorem when $X + Y$ is periodic to give an essentially complete answer to Question 1. We will refer to the exposition of Kemperman's work in the paper [2] of Grynkiewicz, as the definitions used there suit our purposes better than that of the original work [5]. For a detailed discussion of Kneser's Theorem, we refer the reader to Chapter 4 of [7].

2 Preliminaries

Definition 2.1. *Let H be an abelian group. A* small-sum pair*, or* SS-pair *for short, is a pair (X, Y) of subsets of H such that $|X| = |Y| = k$ and $|X+X| = |X+Y| = 2k-1$ for some k.*

Given an abelian group H, subsets $A, B \subseteq H$, and $c \in H$, we define

$$\nu_c(A, B) = |\{(a, b) : a \in A,\ b \in B : a + b = c\}|.$$

If $P \subseteq H$ is a subgroup, we denote by $\phi_P : H \to H/P$ the canonical quotient map.

The *stabilizer* of a subset $A \subseteq H$ is the subgroup of H defined by:

$$\mathrm{Stab}(A) = \{h \in H : h + A = A\}$$

A subset $A \subseteq H$ is called *P-periodic* (or simply *periodic* if the context is clear) if there is a nontrivial subgroup $P \subseteq \mathrm{Stab}(A)$; this is equivalent to requiring that A be a union of P-cosets. A subset $A \subseteq H$ is called *P-subperiodic* if there exists $h \in H - A$ such that $A \cup \{h\}$ is P-periodic.

Definition 2.2. *Let H be an abelian group and $P \subseteq H$ a nontrivial subgroup. A subset $A \subseteq H$ is said to have a* quasi-periodic decomposition *with respect to the* quasi-period P *if there exists a partition of A into two disjoint subsets $A_1 \cup A_0$ such that A_1 is either empty or P-periodic and A_0 is a subset of a P-coset.*

We refer to A_1 as the periodic part *of A and to A_0 as the* aperiodic part *of A. We say that A is* quasi-periodic *if A_1 is nonempty, and that the decomposition is* reduced *if A_0 is not itself periodic.*

Proposition 2.3. *Suppose H is finite, P is a subgroup of H and $A \subseteq H$ is any nonempty subset. If A has a reduced quasi-periodic decomposition with respect to P, such a decomposition is unique.*

Proof.
With notation as above, suppose A has quasi-periodic decompositions $A = A_1 \cup A_0$ and $A = A'_1 \cup A'_0$ with respect to a common quasi-period P. Then A_1,A'_1 are unions of P-cosets, and each of A_0, A'_0 is a proper subset of a single P-coset. It follows that $A_1 = A'_1$ and $A_0 = A'_0$. □

Lemma 2.4. *Suppose H is finite and A, B are subsets of H, and $A = A_1 \cup A_0$ and $B = B_1 \cup B_0$ are quasi-periodic decompositions with respect to a common quasi-period P. Suppose further that neither A nor B is P-periodic. If $A = h + B$, then $A_0 = h + B_0$ and $A_1 = h + B_1$.*

Proof.
If $A = h + B$, then $(h + B_1) \cup (h + B_0)$ is also a quasi-periodic decomposition for A. By Proposition 2.3, it follows that $A_1 = h + B_1$ and $A_0 = h + B_0$. □

The following results are elementary:

Lemma 2.5. *Suppose H is an abelian group and $A \subseteq H$ is a nonempty subset such that $|A + A| = 2|A| - 1$. Then A is not periodic.*

Proof.
Let $k = |A|$. Since (A, A) is an SS-pair, $|A + A| = 2k - 1$. If A is periodic, then there is a nontrivial subgroup $P \subseteq H$ such that A is a disjoint union of P-cosets; in particular, $|P|$ divides k. However, $A + A$ is also a union of P-cosets, so $|P|$ must divide $2k - 1$, which forces P to be the trivial subgroup, a contradiction. □

Lemma 2.6. *Let H be an abelian group. If $A \subseteq H$ is P-periodic, then so is $A + B$ for any $B \subseteq H$.*

Finally, we mention a rudimentary but well-known result:

Lemma 2.7. *[7, Lemma 2.1] Let G be an abelian group and A, B subsets of G such that $|A| + |B| > |G|$. Then $A + B = G$.*

3 Kemperman pairs and resolutions

In this section, we review the theory of critical pairs in abelian groups, following the exposition of Kemperman's work in [2], and prove various results that will help in the solution of our problem.

Definition 3.1. *Let H be an abelian group. A* Kemperman pair *in H is a pair (A, B) of subsets of H such that $A+B$ is not periodic or there exists $c \in H$ such that $\nu_c(A, B) = 1$.*

We now state Kemperman's Theorem [5, Theorem 5.1 and p. 82] in the form given by Grynkiewicz [2].

Theorem 3.2. *([2], p. 563) Let G be an abelian group and (A, B) a Kemperman pair in G. Then $|A + B| = |A| + |B| - 1$ if and only if there exist quasi-periodic decompositions $A = A_1 \cup A_0$ and $B = B_1 \cup B_0$ with nonempty aperiodic parts and common quasi-period P such that (A_0, B_0) is a Kemperman pair and:*

1. *$\nu_c(\phi_P(A), \phi_P(B)) = 1$, where $c = \phi_P(A_0) + \phi_P(B_0)$;*
2. *$|\phi_P(A) + \phi_P(B)| = |\phi_P(A)| + |\phi_P(B)| - 1$, and*
3. *$|A_0 + B_0| = |A_0| + |B_0| - 1$ and the pair (A_0, B_0) is of one of the following (distinct) types:*
    - *(a) $|A_0| = 1$ or $|B_0| = 1$.*
    - *(b) $|A_0| \geq 2$, $|B_0| \geq 2$, and A_0, B_0 are arithmetic progressions with common difference $d \in H$ such that d has order at least $|A_0|+|B_0|-1$. In this case, $\nu_c(A_0, B_0) = 1$ for exactly two values $c \in A_0 + B_0$.*
    - *(c) $|A_0| + |B_0| = |P| + 1$, and there is exactly one element $g \in G$ such that $\nu_g(A_0, B_0) = 1$.*
    - *(d) A_0 is aperiodic, B_0 is of the form $B_0 = g - [(H \setminus A_0) \cap (g' + P)]$ for some $g, g' \in A_0$, and $\nu_c(A_0, B_0) \neq 1$ for all c.*

Lemma 3.3. *Suppose* A, B *are subsets of an abelian group* H *and that* $A = A_1 \cup A_0$, $B = B_1 \cup B_0$ *are quasi-periodic decompositions with respect to some common nontrivial quasi-period* P, *with* $\min\{|A_0|, |B_0|\} \geq 2$. *If there exists an element of* $A + B$ *which has a unique expression as a sum* $a + b$ *with* $a \in A$ *and* $b \in B$, *then* $a \in A_0$ *and* $b \in B_0$. *Hence,* $\nu_c(A, B) = 1 \Leftrightarrow \nu_c(A_0, B_0) = 1$.

Proof.
Suppose towards a contradiction that $a \in A_1$. Select $b_0 \in B_0 \setminus \{b\}$, so that $b - b_0 \in P$. Then the formula $a + b = (a + b - b_0) + b_0$ witnesses that $\nu_{a+b}(A, B) \geq 2$, which is impossible. A similar argument shows that $b \in B_0$. □

The proof of the following is immediate:

Lemma 3.4. *Suppose* A, B *are subsets of an abelian group* H *and that* $A = A_1 \cup A_0$, $B = B_1 \cup B_0$ *are quasi-periodic decompositions with respect to some common quasi-period* P. *If* $A + B$ *is periodic, then* $|A + B| = |\phi_P(A + B)||P|$; *if* $A + B$ *is not periodic, then* $|A + B| = (|\phi_P(A + B)| - 1)|P| + |A_0 + B_0|$.

Lemma 3.5. *Suppose* A *and* B *are subsets of an abelian group* H *with corresponding quasi-periodic decompositions* $A = A_0 \cup A_1$ *and* $B = B_0 \cup B_1$ *with respect to a common quasi-period* P. *Suppose further that* $|A| = |B|$ *and* $|A + B| = |A| + |B| - 1$. *Let* $S = (A_1 + B_1) \cup (A_1 + B_0) \cup (A_0 + B_1)$. *If* $A + B = S$, *then* $A + B$ *is periodic,* $|P| = 2|A_0| - 1$, *and* $|A + B| = |S| = 2|A_1| + |P|$. *If* $A + B \neq S$, *then* $|S| = 2|A_1|$ *and* $|A_0 + B_0| = 2|A_0| - 1$. *In either case,* $|\phi_P(A) + \phi_P(B)| = 2|\phi_P(A)| - 1$.

Proof.
Let $p = |P|$, $x = |A_1|/p = |B_1|/p$, $y = |A_0| = |B_0|$, and $s = |S|/p$. Then $|A + B| = 2xp + 2y - 1$. Also, $1 \leq y \leq |A_0 + B_0| \leq p$. If $s < 2x$, then $|A_0 + B_0| \geq p + 2y - 1 \geq p + 1$, which is impossible, so $s \geq 2x$. Now if $A_0 + B_0 \subseteq S$, then $|A_0 + B_0| = p = 2y - 1$. In this case, then, $A + B$ is periodic, $|P| = 2|A_0| - 1$, and $|A + B| = |S| = (2x + 1)|P| = 2|A_1| + |P|$.

If $A_0 + B_0 \not\subseteq S$ and $s > 2x$, then $|A_0 + B_0| < 2y - 1 - p \leq y - 1$, which is also impossible. So, if $A_0 + B_0 \not\subseteq S$, then $|S| = ps = 2xp = 2|A_1|$ and $|A_0 + B_0| = 2y - 1 = 2|A_0| - 1$.

Finally, $|\phi_P(A)| = |\phi_P(B)| = x + 1$. If $A_0 + B_0 \subseteq S$, then $|\phi_P(A) + \phi_P(B)| = |S|/|P| = 2x + 1$. If $A_0 + B_0 \not\subseteq S$, then $|\phi_P(A) + \phi_P(B)| = |S|/|P| + 1 = 2x + 1$. □

Lemma 3.6. *Suppose* (A, B) *is a critical Kemperman pair in an abelian group* H *and that* $A = A_1 \cup A_0$, $B = B_1 \cup B_0$ *are quasi-periodic decompositions with respect to some common quasi-period* P. *Then* $(\phi_P(A), \phi_P(B))$ *is a critical Kemperman pair* *which is not of type (d). If, moreover, (A, B) is also an SS-pair, then $(\phi_P(A), \phi_P(B))$ is an SS-pair.*

Proof.
Statement (1) of Theorem 3.2 ensures $(\phi_P(A), \phi_P(B))$ is a Kemperman pair which cannot fit the description of type (d) pairs. Now suppose (A, B) is an SS-pair, so $|A| = |B|$. Since A_0 and B_0 are nonempty by Theorem 3.2, elementary divisibility arguments imply that $|A_0| = |B_0|$. Noting that $|A_1| = (|\phi_P(A)| - 1)|P|$, we have $|A| = |A_1| + |A_0| = (|\phi_P(A)| - 1)|P| + |A_0| = (|\phi_P(B)| - 1)|P| + |B_0| = |B|$, so $(|\phi_P(A)| - |\phi_P(B)|)|P| = |B_0| - |A_0| = 0$, and so $|\phi_P(A)| = |\phi_P(B)|$. That $|\phi_P(A)+\phi_P(A)| = 2|\phi_P(A)|-1$ follows from Lemma 3.5 applied to the pair (A, A). □

Lemma 3.7. *Suppose (A, B) is a Kemperman SS-pair in H of type (d). Then $A+B$ is P-subperiodic and A is a translate of B.*

Proof.
As remarked in [2, p. 563], a pair of type (d) satisfies $A_0 + B_0 = (h_0 + P) - \{h_0\}$ for some $h_0 \in H$, which establishes P-subperiodicity of $A + B$. In particular, letting $k = |A_0|$, we have $|P| = |A_0 + B_0| + 1 = |A_0| + |B_0| = 2k$.

By translation, we may assume that A_0 and B_0 are both subsets of P. By the previous paragraph, $A_0 + B_0 \subseteq P$ has $|P| - 1$ elements; moreover, since (A, B) is an SS-pair, $A_0 + A_0$ is also a subset of P of the same order. Hence there exist $x, y \in P$ such that $A_0 + A_0 = P \setminus \{x\}$ and $A_0 + B_0 = P \setminus \{y\}$; setting $B'_0 = B_0 + x - y$, we have $A_0 + B'_0 = P \setminus \{x\}$. Thus, A_0 and B'_0 are subsets of P of order k, each disjoint from $-A_0 + x$. This forces $A_0 = B'_0$, and hence A_0 is a translate of B_0. □

Definition 3.8. *Let (A, B) be a pair of subsets of an abelian group H. We say that (A, B) is* good *if there exist $a \in A$ and $b \in B$ such that $\nu_{a+b}(A, B) = \nu_{a+a}(A, A) = 1$ or* bad *otherwise.*

Clearly, the property of being good or bad is invariant under translation of either set A or B. The next result shows that the qualifiers “good” and “bad” are only meaningful when speaking of pairs of type (c).

Proposition 3.9. *All Kemperman SS-pairs (A, B) of type (a) or (b) are good and all pairs of type (d) are bad.*

Proof.

The statement is clear for pairs of type (a). If (A, B) is of type (b), then A_0 and B_0 are arithmetic progressions of the same length with the same common difference d. By translation, we may assume that $A_0 = B_0 = \{0, d, \ldots, (k-1)d\}$ is a subset of P, where P is a common quasi-period for A and B. Now, $A_0+B_0 = \{0, d, \ldots, 2(k-1)d\}$, and since (A_0, B_0) is a critical pair, the elements $0, d, \ldots, 2(k-1)d$ are distinct, and so $\nu_0(A_0, B_0) = \nu_0(A_0, A_0) = 1$. Since $|A_0| = |B_0| \geq 2$, Lemma 3.3 implies that $\nu_0(A, B) = \nu_0(A, A) = 1$. If (A, B) has type (d), then by Theorem 3.2, there is no $c \in A_0 + B_0$ such that $\nu_c(A_0, B_0) = 1$. This means in particular that $|A_0| = |B_0| \geq 2$, and then Lemma 3.3 implies that there is no $c \in A + B$ such that $\nu_c(A, B) = 1$. Hence, (A, B) is bad. (cf. also [2], (c.8), p. 563.) □

We now give a description of pairs of type (c). If K is a group of odd order, a subset $S \subseteq K$ is called *a set of positives* for K if $0 \in S$, and for every $k \in K$, either $k \in S$ or $-k \in S$ (but not both).

Lemma 3.10. *Suppose (A, B) is an Kemperman SS-pair in H of type (c) and that $A = A_1 \cup A_0$ and $B = B_1 \cup B_0$ are quasi-periodic decompositions with respect to a common quasi-period P. Then $|P| \geq 3$ is odd, $|A_0| = |B_0| \geq 2$, and there exists $h \in H$ such that $A_0 + B_0 = h + P$; in particular, $A + B$ is P-periodic. Furthermore, if (A, B) is good then there exist $h_1, h_2 \in H$ and a set $S \subseteq P$ of positives such that $A_0 = h_1 + S$, $B_0 = h_2 + S$, and $h_1 + h_2 \notin (A_1 + B_1) \cup (A_1 + A_1)$.*

Proof.
By translating A and B appropriately, we may assume that $A_0 \subseteq P$ and $B_0 \subseteq P$. Since $A_0 + B_0$ is a subset of P and $|A_0 + B_0| = |A_0| + |B_0| - 1 = |P|$, it follows that $A_0 + B_0 = P$; moreover, because $|A_0| = |B_0|$, it follows that $|P| = 2|A_0| - 1$ is odd. In particular, since P is nontrivial, $|P| \geq 3$, and so $|A_0| = |B_0| \geq 2$. By Lemma 2.6, A_1+B and A_0+B_1 are both P-periodic; hence, $A+B = (A_1+B)\cup(A_0+B_1)\cup(A_0+B_0)$ is P-periodic.

Now suppose (A, B) is good. By translating each by an appropriate element of P, we may assume further that $\nu_0(A, B) = \nu_0(A, A) = 1$; that is, 0 is representable uniquely as $0 + 0$ both as an element of $A + B$ and as an element of $A + A$. Then $0 \in A_0 \cap B_0$, so A_0 and B_0 are both subsets of P. Then $B_0 \cap (-A_0) = A_0 \cap (-A_0) = \{0\}$; thus, $A_0 = B_0 \subseteq P$ is a set of positives. □

We are now able to give a complete characterization of Kemperman SS-pairs (A, B) for which $A + B$ is periodic:

Proposition 3.11. *Let (A, B) be a Kemperman SS-pair in H together with quasi-periodic decompositions $A = A_1 \cup A_0$ and $B = B_1 \cup B_0$ with respect to a common* *quasi-period P. If (A,B) has type (a) or (d), then $A+B$ is not periodic. If (A,B) has type (c), then $A+B$ is periodic. If (A,B) has type (b), then $A+B$ is periodic if and only if P is cyclic of odd order d, and there exists a generator g of P such that A_0 and B_0 are both translates of $S=\{kg : k=0,1,\dots,\frac{d-1}{2}\}$, which is a set of positives for P.*

Proof.
Clearly $A+B$ is periodic if and only if $|A_0+B_0|=|P|$. For pairs of type (a), $|A_0+B_0|=1$, and since $|P|>1$, $A+B$ is therefore never P-periodic. For pairs of type (d), the conclusion follows from Lemma 3.7, and for pairs of type (c) from Lemma 3.10. The sufficiency of the condition for pairs of type (b) is clear, so suppose (A,B) is a pair of type (b) such that $A+B$ is P-periodic, and (after translating A and B) that A_0 and B_0 are both subsets of P. Then $|P|=|A_0+B_0|=|A_0|+|B_0|-1=2|A_0|-1$, so $|P|$ is odd. Moreover, (A,B) is a good pair by Proposition 3.9, so by translating A and B by an element of P, we may assume $\nu_0(A_0,A_0)=\nu_0(A_0,B_0)=1$. This means that $A_0\cap -A_0=\{0\}$ and $B_0\cap -A_0=\{0\}$. This is only possible if $A_0=B_0$ and A_0 is a set of positives for P. Finally, the common difference $d\in P$ of the arithmetic progressions A_0 and B_0 has order at least $|A_0|+|B_0|-1=|P|$, so P must be cyclic and d a generator for P. □

Proposition 3.12. *Suppose (A,B) is a Kemperman SS-pair in H together with quasi-periodic decompositions $A=A_1\cup A_0$ and $B=B_1\cup B_0$ with respect to a common quasi-period P. Let $C=\phi_P(A)$ and $D=\phi_P(B)$. If (C,D) is of bad type (c), then (A,B) is of bad type (c).*

Proof.
Let $Q\neq 0$ be a common quasi-period of C and D. By translating A and B appropriately, we may assume that $A_0\subseteq P$, $B_0\subseteq P$, and hence that $\phi_P(A_0)=\phi_P(B_0)=0\in C\cap D$. Since (C,D) has bad type and $\nu_0(C,D)=1$ by statement 1 of Theorem 3.2, it follows that $\nu_0(C,C)>1$. Thus, there exists $0\neq c\in C$ such that $-c\in C$. Selecting $a\in A_1$ such that $\phi_P(a)=c$, we have $\phi_P(-a)=-c$ and so $a\in A_1$, $-a\in A_1$, and $A+A$ is P-periodic. Since $|A+A|=|A+B|$, $A+B$ is also P-periodic and hence has type (b) or (c).

Suppose towards a contradiction that (A,B) is of good type. Then we may assume further (by translating by an element of P) that $\nu_0(A,B)=\nu_0(A,A)=1$. Now, (C,D) is of bad type; so arguing as before $\nu_0(C,C)>1$ and there exists $0\neq c\in C$ such that $-c\in C$. Writing $c=\phi_P(a)$ for some $a\in A_1$, choose $a'\neq -a\in A_1$ such that $\phi_P(a') = -c$ (this is possible because $|\phi_P^{-1}(-c)| = |P| > 1$). Thus, $a + a' = p$ for some $p \in P$, $p \neq 0$. Since $a \in A_1$, $a - p \in A_1$, and so $(a - p) + a' = 0$ witnesses that $\nu_0(A, A) \geq 2$, again a contradiction.

Since $A + B$ is P-periodic and has bad type, it must be of type (c) by Propositions 3.9 and 3.11.

□

Remarks.

We note (cf. [2], c.13 on p. 564) that if we select a different quasi-periodic decomposition for A or B, the type of the resulting pair of aperiodic parts does not change; hence we may speak of the type of (A, B) without reference to any choice of quasi-periodic decompositions.

Lemma 3.13. *Let (A, B) be a Kemperman SS-pair in H of type (a), (b), (d), or good type (c) with given quasi-periodic decompositions $A = A_1 \cup A_0$, $B = B_1 \cup B_0$ with respect to a common quasi-period. Then A_0 is a translate of B_0.*

Proof.
For pairs of type (a) or (b), this is clear from the definition. For pairs of good type (c), it follows from Lemma 3.10 and for pairs of type (d) it follows from Lemma 3.7. □

Remark.
If (A, B) is of bad type (c), A_0 may or may not be a translate of B_0.

The following "reconstruction lemma" is rather useful:

Lemma 3.14. *Suppose (A, B) is a Kemperman pair in an abelian group H and $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ are quasi-periodic decompositions with respect to a common quasi-period P. If A_0 is a translate of B_0 and $\phi_P(A)$ is a translate of $\phi_P(B)$ then A is a translate of B.*

Proof.
Note that the conclusion of Lemma 3.14 remains valid if we translate either A or B by any $h \in H$. Moreover, the assertion is automatically true if A_1 and B_1 are empty, so we assume henceforth that these are nonempty. If $\phi_P(B) = \phi_P(A) + \bar{h}$ for some $\bar{h} \in H/P$, then picking $h \in G$ such that $\phi_P(h) = \bar{h}$, we may replace B by $B - h$ and assume henceforth that $\phi_P(B) = \phi_P(A)$. In particular, this means that there exist distinct cosets $h_1 + P, \ldots, h_s + P$ of P in G and decompositions $A = \cup_{i=1}^{s} C_i$ and $B = \cup_{i=1}^{s} D_i$ where for every $i = 1, \ldots, s$, $C_i \subseteq h_i + P$ and $D_i \subseteq h_i + P$ with

$C_i = h_i + P$ if and only if $i \neq i_0$ and $D_j = h_j + P$ if and only if $j \neq j_0$. With this notation, $C_{i_0} = A_0$ and $D_{j_0} = B_0$. We claim that $i_0 = j_0$.

Note that $A+B$ is the union of a P-periodic set, together with $A_0+B_0 \subseteq (h_{i_0}+h_{j_0})+P$. If $i_0 \neq j_0$, then $A_1 + B_1 \supseteq C_{j_0} + D_{i_0} = (h_{i_0} + h_{j_0}) + P \supseteq A_0 + B_0$, so $A + B$ is P-periodic. By Proposition 3.11 $A + B$ cannot be P-periodic when (A, B) is of type (a) or (d), we are reduced to considering pairs (A, B) of type (b) or (c). In this situation, by Theorem 3.2, there exists an element $c \in A_0 + B_0$ such that $\nu_c(A_0, B_0) = 1$. Choose such an element c together with $a \in A_0$ and $b \in B_0$ such that $c = a + b$ and write $a = h_{i_0} + p$, $b = h_{j_0} + p'$ for some $p, p' \in P$. If $i_0 \neq j_0$, then $a + b = (h_{j_0} + p) + (h_{i_0} + p') \in A + B$, so $\nu_c(A, B) \geq 2$. Since $|A_0|, |B_0| \geq 2$ for pairs of type (b) or (c) (cf. Lemma 3.10), Lemma 3.3 implies that $\nu_c(A_0, B_0) \neq 1$, a contradiction.

Thus in all cases, $i_0 = j_0$, and so $A_1 = B_1$. Since $A_0 = p + B_0$ for some $p \in P$, we have that $A = p + B$, as desired. □

Definition 3.15. *Let G be an abelian group and (A, B) a Kemperman pair in G. A* Kemperman resolution *of (A, B) is a pair of families $(\{H_i\}_{i=0}^r, \{(A^{(i)}, B^{(i)})\}_{i=0}^r)$ consisting of the following data:*

- *$G = H_r \supseteq H_{r-1} \supseteq \ldots \supseteq H_1 \supseteq H_0 = 0$ is a collection of distinct subgroups of G. We denote by $\psi_i : G/H_i \to G/H_{i+1}$ the various canonical maps, $i = 0, \ldots, r-1$.*
- *For each $0 = 1, \ldots, r$, $(A^{(i)}, B^{(i)})$ is a Kemperman pair of quasi-periodic subsets of G/H_i with maximum common quasi-period H_{i+1}/H_i with $(A^{(0)}, B^{(0)}) = (A, B)$ and $A_{i+1} = \psi_i(A_i)$, $B_{i+1} = \psi_i(B_i)$.*

Remark.
Theorem 3.2 guarantees that every critical Kemperman pair (A, B) has a Kemperman resolution. Kemperman resolutions, however, are in general not unique.

4 Results

4.1 The Kemperman case

In this section, we answer our question in the case that our original SS-pair (A, B) is a Kemperman pair.

Theorem 4.1. *Let G be a group and (A, B) a Kemperman SS-pair in G such that $|A| = |B|$ and $|A + B| = |A| + |B| - 1$.*

- *If (A,B) is of type (a), (b), or (d), or is of good type (c), then A is a translate of B. In particular, if $A+B$ is not periodic, then A is a translate of B.*
- *If $A+B$ is periodic, then the following are equivalent.*
    1. *A is not a translate of B*
    2. *For every Kemperman resolution $(\{H_i\}_{i=0}^r, \{(A^{(i)}, B^{(i)})\}_{i=0}^r)$ of (A,B), there exists i, $1 \le i \le r$, such that $(A^{(i)}, B^{(i)})$ is a pair of bad type (c) such that $A^{(i)}$ is not a translate of $B^{(i)}$.*
    3. *There exists a Kemperman resolution of (A,B) (as above), together with i, $1 \le i \le r$, such that $(A^{(i)}, B^{(i)})$ is of type (c) and $A^{(i)}$ is not a translate of $B^{(i)}$.*

Proof.
Suppose first that $A+B$ is of type (a), (b), or (d), or of good type (c), and fix a Kemperman resolution $(\{H_i\}_{i=0}^r, \{(A^{(i)}, B^{(i)})\}_{i=0}^r)$ of (A,B). By Lemma 3.12, none of the pairs $(A^{(i)}, B^{(i)})$, $i = 0, \ldots, r$ can be of bad type (c); therefore Lemma 3.13 guarantees that $A_0^{(i)}$ and $B_0^{(i)}$ are translates of each other. Applying Lemma 3.14 inductively to the various homomorphisms $\psi_i : G/H_i \to G/H_{i+1}$, $i = 0, \ldots, r-1$, we conclude that A is a translate of B. If $A+B$ is not periodic, then by Theorem 3.2, (A,B) is of type (a), (b), or (d), so the first assertion follows.

Now suppose that $A+B$ is periodic. If A is a translate of B, then given any Kemperman resolution $(\{H_i\}_{i=0}^r, \{(A^{(i)}, B^{(i)})\}_{i=0}^r)$ of (A,B), $A^{(i)}$ is easily seen to be a translate of $B^{(i)}$, so (3) $\Rightarrow$ (1). The implication (2) $\Rightarrow$ (3) is trivial, so it remains to verify (1) $\Rightarrow$ (2). To this end, suppose $(\{H_i\}_{i=0}^r, \{(A^{(i)}, B^{(i)})\}_{i=0}^r)$ is a Kemperman resolution of (A,B) such that none of the pairs $(A^{(i)}, B^{(i)})$ is of bad type (c). By Lemma 3.13, $A_0^{(i)}$ is a translate of $B_0^{(i)}$ for all $i = 0, \ldots, r$. Applying Lemma 3.14 inductively enables us to conclude that A is a translate of B. □

We now provide a condition on G under which the original question has an affirmative answer without qualification.

Corollary 4.2. *Suppose G is a 2-group and (A,B) is an SS-pair with $|A| = |B|$. Then A and B are translates of each other.*

Proof.
Let $k = |A| = |B|$. First, note that $A+B$ cannot be periodic, because $|A+B| = 2k-1$, which is odd, so it cannot be the union of cosets of a nontrivial subgroup of G. Thus, (A,B) is a Kemperman pair of type (a), (b) or (d), so A is a translate of B by Theorem 4.1. □

4.2 The periodic case

We now study SS-pairs (A, B) which are not Kemperman pairs. For any such pair, $A + B$ is periodic, and in fact, all the results of this section deal with pairs such that $A + B$ is periodic and apply even when (A, B) is a Kemperman pair. In this context, we do not have Theorem 3.2 at our disposal, so we use Kneser's Theorem instead.

Theorem 4.3. *(Kneser, [7, Theorem 4.2])*
Let G be an abelian group and A, B finite nonempty subsets of G; set $H = Stab(A+B)$. If $|A + B| < |A| + |B|$, then

$$|A + B| = |A + H| + |B + H| - |H|$$

In particular, if $|A+B| < |A|+|B|-1$, then H is nontrivial and so $A+B$ is periodic.

It is easy to see that SS-pairs (A, B) such that $\nu_c(A, B) > 1$ for all $c \in A + B$ exist; see Example 3 of Section 5. We describe how, starting with one such pair, one may construct SS-pairs (A, B') where B' is not a translate of A.

Corollary 4.4. *If G is an abelian group and (A, B) a critical pair in G with $P = Stab(A + B)$, then $\phi_P(A + B)$ is aperiodic and $(\phi_P(A), \phi_P(B))$ is a critical pair of subsets of G/P. In particular, if (A, B) is an SS-pair in G, then $(\phi_P(A), \phi_P(B))$ is a Kemperman SS-pair in G/P, and $\phi_P(A)$ is a translate of $\phi_P(B)$.*

Proof.
Aperiodicity is clear in view of the definition of stabilizer. Let $A' = A + P$ and $B' = B + P$; then $\phi_P(A) = \phi_P(A')$, $\phi_P(B) = \phi_P(B')$ and $A' + B' = (A + B) + P = A + B$. By Kneser's Theorem, $|A' + B'| = |A'| + |B'| - |P|$. Dividing all terms by $|P|$, we have $|\phi_P(A + B)| = |\phi_P(A)| + |\phi_P(B)| - 1$. Thus, if (A, B) is an SS-pair in G, then $(\phi_P(A), \phi_P(B))$ is an SS-pair. However, since $\phi_P(A + B) = \phi_P(A) + \phi_P(B)$ is not periodic, $(\phi_P(A), \phi_P(B))$ is a Kemperman pair of type (a), (b), or (d); thus $\phi_P(A)$ is a translate of $\phi_P(B)$ by Theorem 4.1. □

The following Lemma provides a method for constructing new critical pairs out of old ones:

Lemma 4.5. *Let G be an abelian group and A, B subsets of G with $P = Stab(A+B)$. Suppose (A, B) is a critical pair in G, and $A' \subseteq A + P$ and $B' \subseteq B + P$ are any subsets such that $|A'| + |B'| = |A| + |B|$. Then (A', B') is a critical pair in G and $A' + B' = A + B$.*

Proof.
Since (A, B) is a critical pair, Kneser's Theorem implies that $|A|+|B|-1 = |A+B| = |A+P| + |B+P| - |P|$. Moreover, if $A' \subseteq A+P$ and $B' \subseteq B+P$ are *any* two subsets such that $|A'| + |B'| = |A| + |B|$, then $A' + B' \subseteq (A+B) + P = A+B$, so we have $|A'+B'| \leq |A+B| < |A| + |B| = |A'| + |B'|$, so by Kneser's Theorem, $|A'+B'| = |A'+P| + |B'+P| - |P| = |A+P| + |B+P| - |P| = |A+B|$, and so $A'+B' = A+B$. Finally, $|A'+B'| = |A+B| = |A| + |B| - 1 = |A'| + |B'| - 1$, so (A', B') is a critical pair in G. $\square$

Since Question 1 obviously has an affirmative answer if A and B have size 1 or 2, we assume henceforth without further mention that $|A| \geq 3$, $|B| \geq 3$. The next result shows that if $A+B$ is periodic, then there are"many" SS-pairs (A, B) such that A is *not* a translate of B; the only exception occurs when the period is $P = \mathbb{Z}/3\mathbb{Z}$ and A is P-subperiodic. Since Corollary 4.4 implies that $\phi_P(A)$ is always a translate of $\phi_P(B)$, we assume $\phi_P(A) = \phi_P(B)$ to simplify the discussion.

Theorem 4.6. *Suppose G is an abelian group and $A \subseteq G$ is a subset such that $|A+A| = 2|A| - 1$ and $P = Stab(A+A) \neq 0$. Consider the collection $\mathcal{B}$ of all subsets $B \subseteq G$ such that $|A| = |B| = k$, $\phi_P(A) = \phi_P(B)$, and (A, B) is an SS-pair. Then $|\mathcal{B}| = \binom{|A+P|}{|A|}$ and at most $|A+P|$ members of $\mathcal{B}$ are translates of A. In particular, there are at least $m = \binom{|A+P|}{|A|} - |A+P|$ subsets $B \subseteq G$ such that (A, B) is an SS-pair and B is not a translate of A. Moreover, $m = 0$ if and only if $P \cong \mathbb{Z}/3\mathbb{Z}$ and $|A|$ is P-subperiodic.*

Proof.
Since $\phi_P(A) = \phi_P(B)$, any $B \in \mathcal{B}$ must satisfy $B \subseteq A+P$. Observe that by Lemma 2.5, A must be a strict subset of $A+P$. Moreover, if $B \subseteq A+P$ is any (necessarily strict) subset such that $|A| = |B|$, then (A, B) is an SS-pair by Lemma 4.5. Hence $|\mathcal{B}| = \binom{|A+P|}{|A|}$ and, since a translation is determined uniquely by where it sends a fixed element, at most $|A+P|$ such subsets $B \subseteq A+P$ could be translates of A.

It is clear that $m = 0$ is possible only if $|A| = |A+P|-1$; that is, A is P-subperiodic. In this case, there are integers c, d such that $|A| = c|P| - 1$ and $|A+A| = d|P|$. This implies $d|P| = |A+A| = 2|A| - 1 = 2(c|P| - 1) - 1$; thus, $(2c - d)|P| = 3$ and so $|P| = 3$. Conversely, if $|P| = 3$ and A is P-subperiodic, it is clear that any subset $B \subseteq A+P$ of size $|A|$ must be a translate of A, and so $m = 0$. $\square$

5 Examples

In this section we give some examples to illustrate the phenomena described in previous sections. The first two examples involve Kemperman pairs and are analyzed using the techniques of Section 4.1, while the third example studies a non-Kemperman pair using the methods of Section 4.2.

Example 1.
Let $G = \mathbb{Z}/8\mathbb{Z}\times\mathbb{Z}/11\mathbb{Z}$, and let $A = \{(0,0),(1,0),(2,0),(4,0)\}\cup \mathbb{Z}/8\mathbb{Z}\times\{0,1,5,7,8,9\}$, $B = \{(1,0),(2,0),(3,0),(5,0)\} \cup\ \mathbb{Z}/8\mathbb{Z}\times\{0,1,5,7,8,9\}$.
One checks readily that $A+B = G - \{(0,0\}$. Thus, $|A+B| = 87$; since $A+B$ cannot be periodic, (A,B) is a Kemperman pair. Since clearly $B = A+(1,0)$, we have that $|A+A| = 87$ and so (A,B) is also an SS-pair in G. Following the notation of Definition 3.15, we define $H_0 = 0,\ H_1 = \mathbb{Z}/8\mathbb{Z}\times\{0\},\ H_2 = G$. We then may describe a Kemperman resolution of (A,B), using the following sequence as an aid:

$$\mathbb{Z}/8\mathbb{Z}\times\mathbb{Z}/11\mathbb{Z} \overset{\psi_0}{\to} \mathbb{Z}/11\mathbb{Z} \overset{\psi_1}{\to} 0$$

We describe the quasi-periodic decompositions of the various pairs $(A^{(i)}, B^{(i)})$ as follows; clearly it suffices to describe the pairs $(A_0^{(i)}, B_0^{(i)})$.

$$A_0 = A_0^{(0)} = \{0,1,2,4\},\ B_0 = B_0^{(0)} = \{1,2,3,5\} \subseteq \mathbb{Z}/8\mathbb{Z}$$

Then $A^{(1)} = \psi_0(A) = \{0,1,3,5,7\} = \psi_0(B) = B^{(1)} \subseteq \mathbb{Z}/11\mathbb{Z}$, and we take $A_0^{(1)} = A^{(1)}$, $B_0^{(1)} = B^{(1)}$. In this example, $(A_0^{(1)}, B_0^{(1)})$ is a pair of (good) type (c), and (A_0, B_0) is a pair of (bad) type (d).

Example 2.
Let $G = \mathbb{Z}/35\mathbb{Z}\times\mathbb{Z}/25\mathbb{Z}$, and let

$$A = \{(0,0),(0,5),(0,10)\}\cup\{(7,5k) : 0\le k\le 4\}\cup\{(14,5k) : 0\le k\le 4\}\cup$$
$$\{(5i+j, 5k+2) : 0\le i\le 6, 0\le j\le 4, 0\le k\le 4\},$$

$$B = \{(0,1),(0,6),(0,16)\}\cup\{(7,5k+1) : 0\le k\le 4\}\cup\{(14,5k+1) : 0\le k\le 4\}\cup$$
$$\{(5i+j, 5k+3) : 0\le i\le 6, 0\le j\le 4, 0\le k\le 4\}.$$

Then $\nu_{(21,0)}(A,B) = 1$, so (A,B) is a Kemperman pair. We describe a Kemperman resolution:

$$\mathbb{Z}/35\mathbb{Z}\times\mathbb{Z}/25\mathbb{Z} \overset{\psi_0}{\to} \mathbb{Z}/35\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z} \overset{\psi_1}{\to} \mathbb{Z}/5\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z} \overset{\psi_2}{\to} \mathbb{Z}/5\mathbb{Z} \overset{\psi_3}{\to} 0$$

where

$$H_0 = 0,\ H_1 = \{(0, 5k) : 0 \le k \le 4\},$$

$$H_2 = \{(7j, 5k) : 0 \le j \le 6, 0 \le k \le 4\}$$

$$H_3 = \{(m, 5k) : m \in \mathbb{Z}/35\mathbb{Z},\ 0 \le k \le 4\}$$

$$H_4 = G$$

The quasi-periodic decompositions of the various pairs $(A^{(i)}, B^{(i)})$ are given below; as above, it suffices to describe $(A_0^{(i)}, B_0^{(i)})$.
For $(A, B) = (A^{(0)}, B^{(0)})$, we have
$A_0^{(0)} = \{(0, 0), (0, 5), (0, 10)\}$, $B_0^{(0)} = \{(0, 1), (0, 6), (0, 16)\}$. Then

$$A^{(1)} = \psi_0(A) = \{(0, 0), (7, 0), (14, 0)\} \cup \{(5i + j, 2) : 0 \le i \le 6,\ 0 \le j \le 4\} \text{ and}$$

$B^{(1)} = \psi_0(B) = \{(0, 1), (7, 1), (14, 1)\} \cup \{(5i+j, 3) : 0 \le i \le 6,\ 0 \le j \le 4\}$ and we choose $A_0^{(1)} = \{(0, 0), (7, 0), (14, 0)\}$, $B_0^{(1)} = \{(0, 1), (7, 1), (14, 1)\}$. Then

$A^{(2)} = \psi_1(A^{(1)}) = \{(0, 0)\} \cup \mathbb{Z}/5\mathbb{Z} \times \{(2)\}$ and $B^{(2)} = \psi_1(B^{(1)}) = \phi_{H_3}(B) = \{(0, 1)\} \cup \mathbb{Z}/5\mathbb{Z} \times \{(3)\}$

and we choose $A_0^{(2)} = \{(0, 0)\}$, $B_0^{(2)} = \{(0, 1\}$.
Finally, $A^{(3)} = \psi_2(A^{(2)}) = \{0, 2\}$, $B^{(3)} = \psi_2(B^{(2)}) = \{1, 3\}$, and we choose $A_0^{(3)} = A^{(3)}$, $B_0^{(3)} = B^{(3)}$.

In this example, $(A_0^{(3)}, B_0^{(3)})$ has type (b), $(A_0^{(2)}, B_0^{(2)})$ has type (a), $(A_0^{(1)}, B_0^{(1)})$ has type (b), and $(A_0^{(0)}, B_0^{(0)})$ has (bad) type (c). Note that $A^{(3)} + B^{(3)}$ is not periodic, whereas $A^{(1)} + B^{(1)}$ is, even though both are of type (b).

Example 3.
Let $G = \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, and $A = B = \{(0, 0), (0, 1), (0, 2), (3, 0), (6, 0)\}$. Then $|A| = |B| = 5$, and by Lemma 2.7, $A + B = H = \{0, 3, 6\} \times \mathbb{Z}/3\mathbb{Z}$, so (A, B) is an SS-pair; however, $\nu_c(A, B) > 1$ for all $c \in A + B$. Since $Stab(A + B) = H$, we may use Lemma 4.5 to construct SS-pairs (A, C) for which C is not a translate of A by choosing C be any five element subset of H which is not a translate of A; there are at least $\binom{9}{5} - 9 = 117$ of these.

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