NOTES ON TODORCEVIC'S ERICE LECTURES ON FORCING WITH A COHERENT SUSLIN TREE

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1. Part I

1.1. **The P-ideal dichotomy.** The P-ideal dichotomy is the statement that whenever I is a P-ideal on a set X, either X is a countable union of sets orthogonal to I (i.e., intersecting no member of I infinitely), or there is an uncountable subset of X whose countable subsets are all in I. The statement is not weakened when we assume that I consists of countable sets, which we do here.

First we review a proper forcing which forces an instance of the P-ideal dichotomy. Let X and I be as above. Let $\kappa = (|X|^{\aleph_0})^+$. For each countable elementary submodel M of $H(\kappa)$ with X and I in M, fix an element a_M of I which contains mod finite all members of $M \cap I$. Let P be the partial order whose conditions p are pairs $(\mathcal{M}_p, Y_p)$, where $\mathcal{M}_p$ is a finite $\in$-chain of elementary submodels of $H(\kappa)$ with X and I as members, and Y_p is a finite $\mathcal{M}$-separated (i.e., for any two members of Y_p there is an element of $\mathcal{M}_p$ that has one as an element and not the other) subset of X such that for each $y \in Y_p$ and each $M \in \mathcal{M}_p$, if $y \in M$ then $y \in a_M$, and if $y \notin M$ then y is not in any set in M orthogonal to I. The order is inclusion on both coordinates.

Now suppose that p is a condition, and N is an elementary submodel of $H((2^{|P|})^+)$ with P and p as elements. Let p' be the condition $(\mathcal{M}_p \cup \{N \cap H(\kappa)\}, Y_p)$. We want to see that p' is (P, N)-generic. So let D be a dense subset of P in N and let r be a condition below p'. We may assume that $r \in D$. Let M_0 be the largest model in $\mathcal{M}_r \cap N$.

Arguing in N, and identifying finite subsets of X with their increasing enumeration in terms of some wellordering of X in all models of $\mathcal{M}$, let $\mathcal{T}$ be the tree of finite increasing sequences t from X such that

- all members of t are greater than all members of $Y_r \cap N$,
- no member of t is in any set in M_0 orthogonal to I,
- there is an extension Z of $(Y_r \cap N)^\frown t$ of length $|Y_r|$ for which there is some condition $q \in D$ with $Y_q = Z$.

Note that $Y_r \setminus N$ is in $\mathcal{T}$. Now thin $\mathcal{T}$ (iteratively removing as few nodes as possible) to a tree $\mathcal{T}'$ such that for each node t of $\mathcal{T}'$ of length less than $|Y_r \setminus N|$ (including the emptyset), the set of $x \in X$ such that $t^\frown\langle x\rangle \in \mathcal{T}'$ is not orthogonal to I. This thinning takes $|Y_r \setminus N|$ many rounds starting, one for each non-terminal level of the tree, proceeding from the top down. Note that $Y_r \setminus N$ is still in $\mathcal{T}'$, since for each proper initial segment t of $Y_r \setminus N$, t is in some elementary submodel M of

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$H(\kappa)$, and the next element of Y above the maximum of t is not in any set in M orthogonal to I.

Now we can choose a cofinal branch through $\mathcal{T}'$ consisting of elements of N, with the property that the elements of the branch are all in a_M for all $M \in \mathcal{M}_r \setminus N$. To see this, note that at each point of our construction the set of possible extensions in $\mathcal{T}'$ must contain an infinite element of I, and all but finitely many of the members of I will be in $\bigcap_{M \in \mathcal{M} \setminus N} a_M$.

This completes the proof.

1.2. **PFA(S) and the P-ideal dichotomy.** Now suppose that S is a coherent Suslin tree, λ is a cardinal and $\dot{I}$ is an S-name for a P-ideal on λ such that λ is not a countable union of sets orthogonal to $\dot{I}$. Again, let κ be $(\lambda^{\aleph_0})^+$. For each countable elementary submodel M of $H(\kappa)$ with S and $\dot{I}$ as members, we choose a name $\dot{a}_M$ for a countable subset of λ such that the members of $S_{\omega_1 \cap M}$ (where S_α denotes the αth level of S) decide $\dot{a}_M$, and the realization of $\dot{a}_M$ is forced to

- contain mod finite all members of the realization of $\dot{I}_M$.
- be contained mod finite in some member of I containing mod finite all members of the realization of $\dot{I}_M$,

where $\dot{I}_M$ is the name for the realization of all the names in M for members of I. We can find such a name by filling an appropriate (ω, ω)-gap corresponding to each member of $S_{\omega_1 \cap M}$. Since we assume that $\dot{I}$ is a name for an ideal containing all finite subsets of λ, $\dot{a}_M$ is in fact a name for a member of the realization of $\dot{I}$.

For each such M, let $\dot{\xi}_M$ be the canonical (nice) name for the least element of λ not in any subset of λ orthogonal to $\dot{I}$ realized by a name in M.

We now define the forcing P. A condition in p is a function whose domain is a finite $\in$-chain $\mathcal{M}_p$ of countable elementary submodels of $H(\kappa)$ with S and $\dot{I}$ as members, and range contained in S, such that for each $M \in \mathcal{M}_p$, $p(M)$ is not in M but is in all members of $\mathcal{M} \setminus M$, and that $p(M)$ decides the value of $\dot{\xi}_M$ (note that $p(M)$ will also decide the value of $\dot{a}_M$, though this is less important). The function p must have the further property that if M, N are in $\mathcal{M}_p$ and $p(N) < p(M)$, then $p(M)$ forces that $\dot{\xi}_N \in \dot{a}_M$.

Now suppose that p_0 is a condition in P, and N is a countable elementary submodel of $H((2^{|P|})^+)$ with P and p_0 as members. Let p_1 be the condition $p_0 \cup \{(N \cap H(\kappa), t')\}$, where t' is any element of $S \setminus N$. Now let s_0 be any element of $S_{N \cap \omega_1^M}$. We need to see that (p_1, s_0) is $(P \times S, N)$-generic.

Let (r, s_1) be an element of $P \times S$ below (p_1, s_0). We may assume that $(r, s_1) \in D$, and that the height of s_1 is greater than the height of any member of the range of r. Fix $\gamma_0 \in \omega_1 \cap N$ such that no member of the range of r disagrees with s_0 at any point in the interval $[\gamma_0, \omega_1 \cap N)$. Enumerate the models of the domain of r (as ordered by the $\in$-relation) as $\langle Q_i : i < |r| \rangle$.

For any condition $p \in P$, let Ξ_p be the function with the same domain as p where $\Xi_p(Q)$ is the value of $\dot{\xi}_Q$ as decided by $p(Q)$.

For each $t \in T$, let $\mathcal{T}_t$ be the tree of consisting of all initial segments of increasing sequences e from X which are the ranges of $\Xi_{p \setminus (r \cap N)}$, for some $p \in P$ end-extending $(r \cap N)$ such that $(p, t) \in D$, $|p| = |r|$ and

- for each $i < |r|$, if M is the the ith element of the domain of p, then $p(M)$ agrees with $r(Q_i)$ up to γ_0, and $p(M)$ agrees with t after γ_0 if and only if $r(Q_i)$ agrees with s_1 after γ_0.

Let a be the set of $i < |r|$ such that $r(Q_i)$ does not disagree with s_1 on ordinals greater than or equal to γ_0.

Since D is closed under strengthening the right coordinate, $\mathcal{T}_t \subseteq \mathcal{T}_{t'}$ whenever $t \geq_S t'$.

For each $t \in S$, thin $\mathcal{T}_t$ to a tree $\mathcal{T}_t'$ (iteratively removing as few nodes as possible, level by level) such that for each $\sigma \in \mathcal{T}_t$ (including the empty sequence),

- if
  - $|r \cap N| + |\sigma| + 1 \in a$,
  - B_σ is the set of immediate successors of σ in $\mathcal{T}_t'$,

  then B is forced by the union of t beyond γ_0 with $r(Q_{|r \cap N| + |\sigma| + 1}) \restriction \gamma_0$ to have infinite intersection with some countable set C forced by this condition to be in $\dot{I}$ (which since this union is M-generic is the same as saying that the union does not force B to be orthogonal to $\dot{I}$).

Claim. *The range of* $\Xi_{r \setminus N}$ *is in* $\mathcal{T}_{s_1}$

Proof : For each $t \in S$, let $\mathcal{T}_t^0 = \mathcal{T}_t$. For each ordinal $j < |r \setminus N|$ and each $t \in S$, $\mathcal{T}_t^{j+1}$ is formed from $\mathcal{T}_t^j$ by thinning removing those sequences from $\mathcal{T}_t^j$ of length $|r \setminus N| - j - 1$ whose set of immediate successors is not sufficiently large. It suffices then to fix $j < |r \setminus N|$, to suppose that the range of $\Xi_{r \setminus N}$ is in $\mathcal{T}_{s_1}^j$ and show that it is in $\mathcal{T}_{s_1}^{j+1}$. To do this, let $i = |r \setminus N| - j - 1$, and let σ_i be the first i many member of the range of $\Xi_{r \setminus N}$.

Let U be the set of $t \in S$ such that $\sigma_i \in \mathcal{T}_t^j$. Then $U \in Q_{i+1}$.

For each $t \in U$, let B_σ^t be the set of immediate successors of σ_i in $\mathcal{T}_t^j$.

If there exist $t \geq_S t'$ in $S \cap Q_{i+1}$ below s_1 such that t' forces B_t not to be orthogonal to I, then we are done. Otherwise, there is a name in Q_{i+1} for the union of the sets B_σ^t along the generic branch, and this set must be forced by some initial segment of s_1 in Q_{i+1} to be orthogonal as it is an increasing union of uncountably many orthogonal sets. But s_1 forces that $\Xi_r(Q_{i+1})$ is not in this set, and $\Xi_r(Q_{i+1}) \in B_\sigma^{s_1}$, giving a contradiction. This concludes the proof of the claim.

Then $\mathcal{T}_{s_1}$ has height $|r|$, so the set of $s \in S$ extending $s_1 \restriction \gamma_0$ for which $\mathcal{T}_s$ has height $|r|$ contains s_1, so we can find such an s_2 in N which is an initial segment of s_1. Then we can find a condition of size $|r|$ in $N \cap \mathcal{T}_{s_2}$ such that the corresponding ξ's are in the required realizations of the names $\dot{a}_M$, minus their finite errors. We do this by finding in N a branch p_2 (i.e., p_2 is the set of left-coordinates of the branch) through $\mathcal{T}_{s_2}$ with the property that for each $M \in \mathcal{M}_{p_2}$ and each $Q \in \mathcal{M}_r \setminus N$, if $p_2(M) < r(Q)$, then $\dot{\xi}_M$ as decided by $p_2(M)$ is in the set $\dot{a}_Q$ as decided by $r(Q)$. Note that as we do this, if $i < |r|$ and $r(Q_i)$ disagrees with s_1 above γ_0, then the same will be true for the ith level of p_2, so the hypotheses of the above implication will not be satisfied. In the other case, the set of values $\dot{\xi}_M$ for potential models M at the ith level (according to $\mathcal{T}_{s_2}$) is forced by the union of s_2 beyond γ_0 with $r(Q_i) \restriction \gamma_0$ to have infinite intersection with some countable set C forced by this condition to be in $\dot{I}$. Then for each $Q \in r \setminus N$ such that $r(Q)$ agrees with $r(Q_i)$ up to γ_0, $\dot{a}_N$ (as decided by $s_1 \restriction [\gamma_0, N \cap \omega_1] \cup r(Q_i) \restriction \gamma_0$ contains all but finitely much of C, so there is some member of $C \cap B$ which is in all of these sets. Choose the ith model M of P_2 so that the realization of $\dot{\xi}_M$ is such a member. This completes the proof that (p_1, s_0) is $(P \times S, N)$-generic.

Finally, let us suppose that $\langle M_\alpha : \alpha < \omega_1 \rangle$ is a generic sequence for P, with a corresponding function p whose domain is this sequence and whose range is contained in S. The set of conditions $p(M_\alpha)$ is somewhere dense in S, and any branch through S below this condition will force that the realizations of the names $\dot{\xi}_{M_\alpha}$ for which $p(M_\alpha)$ is in the generic branch will be an uncountable set whose countable subsets are all in the realization of $\dot{I}$. Since we could carry out this entire argument below any node of S, a dense set of nodes in S force the existence of such an uncountable set and this completes the proof that under PFA(S) the P-ideal dichotomy holds after forcing with S.

2. Part II

The following lemma appears on page 27 of the slides from the lectures.

Lemma 2.1. *For a subset X of a compact space K, the U-sequential closure of X in K is the set*

$$\bar{X}^U = \{ \lim_{b \to U_\alpha} x_n : \alpha < \omega_1, \{x_n : n \in \omega\} \subseteq X \}.$$

Now we aim to prove the following.

Theorem 2.2. *The coherent Suslin tree S forces that if $\dot{K}$ is a compact countably tight space then for every ground model ultrafilter U on ω, U-sequentially closed subsets of $\dot{K}$ are closed.*

To prove this, we suppose that S forces that $\dot{Z}$ is a U-sequentially closed non-closed subset of some compact space $\dot{K}$, whose underlying space is the cardinal λ. We will show that S forces that $\dot{K}$ is not countably tight. Since the realization of $\dot{Z}$ is U-sequentially closed, it is countably compact, so it is not Lindelöf. We fix names $V_{\dot{x}}$ and $U_{\dot{Z}}$ for names $\dot{x}$ for elements of $\dot{Z}$, in such a way that the following are forced to hold.

- $\dot{x} \in V_{\dot{x}} \subseteq \bar{V}_{\dot{x}} \subseteq U_{\dot{x}}$,
- $\dot{Z} \setminus \bigcup_{x \in \dot{Z}_0} U_x \neq \emptyset$ for all names $\dot{Z}_0$ for countable subsets of $\dot{Z}$.

We will produce a proper poset P which preserves S and which forces the existence of a sequence $\langle \dot{x}_\xi : \xi < \omega_1 \rangle$ of S-names for elements of $\dot{Z}$ such that for all $\alpha < \omega_1$, S forces that

- $\{\dot{x}_\xi : \xi \leq \alpha\} \subseteq V_{\dot{x}_\alpha}$,
- $\{\dot{x}_\xi : \xi > \alpha\} \subseteq \dot{Z} \setminus U_{\dot{x}_\alpha}$.

This will imply that $\dot{K}$ is not countably tight.

We let $\dot{F}$ be an S-name for a collection of U-sequentially closed subsets of $\dot{Z}$ such that

- $\dot{F}$ is forced to contain all sets of the form $\dot{Z} \setminus \bigcup_{x \in Z_0} U_x$, for Z_0 a countable subset of $\dot{Z}$,
- the intersection of any finitely many members of $\dot{F}$ is infinite (equivalently, nonempty, by the previous property),
- $\dot{F}$ is maximal with respect to these two properties.

Our forcing notion will involve countable elementary substructures of some $H(\theta)$ ($(2^{2^\lambda})^+$?) with a fixed wellordering $<_\theta$. We fix a sequence of bijections between ω and each countable ordinal, and let $\{\dot{x}^N_i : i < \omega\}$ be the ordering of (the S-names for elements of $\dot{F}$ in) N induced by this sequence and $<_\theta$.

For each such N, and each $k \in \omega$, let $\dot{y}^N_k$ be the S-name for the least element of

$$\bigcap(\dot{F} \cap \{x^N_i : i \leq k\}),$$

if this intersection is nonempty, and any point of $\dot{Z}$ otherwise. Then for each N the name $\dot{x}_N$ for the U-limit of $\langle \dot{y}^N_k : k < \omega\rangle$ is well defined, and we let $\dot{V}_N = V_{\dot{x}_N}$ and $\dot{U}_N = U_{\dot{x}_N}$.

Now we define our forcing P. Conditions in P are functions p whose domains are finite $\in$-chains $\mathcal{N}_p$ of countable elementary submodels of $\langle H(\theta), \in, <_\theta\rangle$ such that for each $M \in \mathcal{N}_p$, $p(M)$ is a pair $(p_0(M), p_1(M))$, where

- $p_0(M) \in S \setminus M$,
- $p_1(M)$ is an ordinal less than $\omega_1 \cap M$,
- $p_0(M)$ decides the value of $\dot{x}_M$, and all $\dot{y}^M_k$,
- $p_0(M)$ decides all statements of the forms $\check{\alpha} \in V_{\dot{x}_M}$, $\check{\alpha} \in U_{\dot{x}_M}$, for $\alpha \in \lambda \cap M$.

We have the following additional requirements when M and N are in $\mathcal{N}_p$ and $M \in N$.

- $p_0(M) \in N$,
- if $p_0(M) < p_0(N)$ and $M \cap \omega_1 > p_1(N)$, then $p(N)$ forces that $\dot{x}_M \in V_{\dot{x}_N}$.

The order is inclusion. We let X_p and Y^k_p ($k \in \omega$) be the functions with the same domain as p such that $X_p(M)$ and $Y^k_q(M)$ are the respective values of $\dot{x}_M$ and $\dot{y}^k_M$ as decided by $p_0(M)$.

We first show that $P \times S$ is proper. This amounts to showing that if

- $D \subset P \times S$ is dense,
- M is a a countable elementary submodel of $H((2^{|P|})^+)$,
- $(q^0, t^0) \in D$, $M \cap H(\theta) \in \mathcal{N}_q$,
- $t^0 \in S \setminus M$,

there exists a (q^1, t^1) in $D \cap M$ compatible with (q^0, t^0).

We may assume that t^0 has height greater than all elements of the range of q^0. For notational convenience, we assume that $q^0 \cap M$ and $q^0 \setminus M$ are nonempty. We let $\langle Q^0_i : i < |q_0|\rangle$ denote the elements of $\mathcal{N}_{q^0}$ in the order given by $\in$. Let $\gamma_0 > q^0_1(M \cap H(\theta))$ be an ordinal in $M \cap \omega_1$ such that all elements of the range of q^0_0 agree with t^0 on the interval $[\gamma_0, \omega_1 \cap M)$. Let a be the set of $i < |q_0|$ such that $q^0_0(Q^0_i)$ agrees with t^0 on ordinals greater than or equal to γ_0.

For each $t \in S$ such that $t \restriction \gamma_0 = t^0 \restriction \gamma_0$, we let $\mathcal{T}_t$ be the tree of initial segments of ranges of functions $X_{q \setminus M}$, for q a condition in P (with domain $\langle Q_i : i < |q|\rangle$) such that

- $q^0 \cap M$ is an initial segment of q,
- $|q| = |q^0|$,
- for all $i < |q|$, $q_0(Q_i) \restriction \gamma_0 = q^0_0(Q^0_i) \restriction \gamma_0$, and $q_0(Q_i)$ agrees with t on ordinals greater than or equal to γ_0 if and only if $i \in a$,
- $(q, s) \in D$.

Note that the range of X_{q^0} is in $\mathcal{T}_{t^0}$.

Given a tree T and a node σ, we let $IS(\sigma, T)$ denote the immediate successors of σ in T.

Iteratively thin each $\mathcal{T}_t$ (in $|q^0 \setminus M| - 1$ stages, level by level, removing as few nodes as possible) to a tree $\mathcal{T}'_t$ meeting the following condition. Suppose that $\sigma \in \mathcal{T}'_t$ and $|\sigma + 1| \in a$. We require then that the node of S which agrees with $q^0_0(Q^0_{i+1})$ below γ_0 and with t thereafter forces that the U-sequential closure of $\bigcup_{t' \in G} IS(\sigma, \mathcal{T}'_{t'})$ is an element of $\dot{F}$, where G denotes the generic branch through S.

The trees are refined in $|q^0 \setminus M| - 1$ many stages. Let $\mathcal{T}^0_t = \mathcal{T}_t$. The jth stage takes us from $\mathcal{T}^j_t$ to $\mathcal{T}^{j+1}_t$. At this stage nodes of length $|q^0 \setminus M| - j - 1$ are removed. Then $\mathcal{T}'_t = \mathcal{T}^{|q^0 \setminus M|}_t$.

Whenever $t \geq_S t'$ and $j < |q^0 \setminus M| - 1$, $\mathcal{T}^j_t \subseteq \mathcal{T}^j_{t'}$.

Let us see that the range of X_{q^0} remains in $\mathcal{T}'_{t^0}$. Let σ be the first initial segment of the range of X_{q^0} removed. Let $i = |\sigma|$, let $j = |q^0 \setminus M| - j - 1$, and note that σ was removed in passing from $\mathcal{T}^j_{t_0}$ to $\mathcal{T}^{j+1}_{t_0}$. Let π be the map which moves nodes of S above $t^0 \restriction \gamma^0$ to the corresponding nodes above $q^0_0(Q^0_{i+1}) \restriction \gamma_0$.

If there is a node $t \leq_S t^0$ such that $\sigma \in \mathcal{T}^{j+1}_t$ then we are done. Otherwise, there is a $t' \leq t^0$ in Q^0_{i+1} such that the U-sequential closure of $\bigcup_{t' \in G} IS(\sigma, \mathcal{T}^j_t)$ is forced by $\pi(t)$ not to be an element of $\dot{F}$. Then (by the maximality of $\dot{F}$) there is a name $\dot{C}$ for an element of $\dot{F}$ forced by $\pi(t)$ to be disjoint from the U-sequential closure of $\bigcup_{t' \in G} IS(\sigma, \mathcal{T}^j_t)$. Then $\dot{C}$ is forced to be U-sequentially closed and to contain all but finitely many of the $y^{Q^0_{i+1}}_k$'s, which means that $\dot{x}_{Q^0_{i+1}}$ is also forced to be in $\dot{C}$. However, $t_0 \leq_S t'$, and $X_{q \setminus M}(Q^0_{i+1}) \in IS(\sigma, \mathcal{T}^j_{t^0})$. Since $X_{q \setminus M}(Q_{i+1})$ is the value of $\dot{X}_{Q^0_{i+1}}$ decided by $q^0_0(Q^0_{i+1})$, this gives a contradiction.

It follows that there is an initial segment t^1 of t^0 in $S \cap M$ such that $\mathcal{T}'_{t^1}$ has height $|q^0|$. We now recursively pick ordinals ξ_j and nodes t^1_j of S for $j < |q^0 \setminus M|$ such that $t^1 = t^1_0 \geq_S \ldots \geq_S t^1_{|q^0 \setminus M|} \geq_S t^0$ and $\sigma_{j+1} = \langle \xi_0, \ldots, \xi_j \rangle \in \mathcal{T}'_{t_j}$ for all $j < |q^0 \setminus M|$ (we let σ_0 denote the empty sequence).

If $|\sigma| + 1$ is not in a, we can pick any successor of σ in $\mathcal{T}'_{t^1_j}$ and let $t^1_{j+1} = t^1_j$. So suppose that $|\sigma| + 1 \in a$.

Let E be the set of $Q \in \mathcal{N}_{q^0} \setminus N$ such that $q^0_1(Q) < M \cap \omega_1$ and $q^0_0(Q) \restriction \gamma_0 = q_0(Q^0_{i+1}) \restriction \gamma_0$. For each such Q, $q^0_0(Q)$ forces that $\dot{x}_M \in V_{\dot{x}_N}$. Then for each $Q \in E$ there is a set $f_Q \in U$ such that $q_0(Q)$ forces that $\dot{y}^M_k \in q_1(Q)$ for all $k \in f_Q$.

Fix y which is equal to y^M_k for some k in the intersection of these f_Q's, such that y is also in the U-sequential closure of $IS(\sigma, \mathcal{T}'_{t^1_{j+1}})$ for some $t^1_{j+1} \leq_S t^1_j$ such that $t^1_{j+1} \geq_S t^0$, using the fact that t^1_j forces that the U-sequential closure of $\bigcup_{t' \in G} IS(\sigma, \mathcal{T}'_{t'})$ is in $\dot{F}$, and that this is equal to the union over all $t' \in G$ of the U-sequential closures of the sets $IS(\sigma, \mathcal{T}'_{t'})$. Then there is an $\alpha \in M \cap \omega_1$ such that y is the U_α-limit of a sequence $\langle z_i : i < \omega \rangle$ of points in $IS(\sigma, \mathcal{T}'_{t^1_{j+1}})$. Then again there are sets $g_Q \in U_\alpha$ for each $Q \in E$ such that for each $k \in g_Q$, $q^0_0(Q)$ forces that $z_k \in \dot{V}_Q$. So we can pick an i in the intersection of these g_Q's and use the corresponding $z_i \in IS(\sigma, \mathcal{T}'_{t^1_{j+1}})$ to pick ξ_j. This completes the proof of the properness of $P \times S$.

Finally, let us see that P forces that S forces that $\dot{K}$ is not countably tight. Forcing with P gives an $\in$-chain N_ξ $(\xi \in E)$ of elementary submodels of $H(\theta)$ such that the following hold, where $t_\xi = p(N_\xi) \in S \setminus N_\xi$ (for some p in the generic filter with N_ξ in its domain) is the node of S associated to N_ξ by the generic filter:

(1) E is a stationary subset of ω_1,
(2) $\eta = sup(E \cap \eta)$ implies $N_\eta = \bigcup_{\xi \in E \cap \eta} N_\xi$,
(3) for all $\xi < \eta$ in E,
  - $N_\xi \in N_\eta$,
  - $t_\xi \in N_\eta$,
  - t_η forces that $\dot{x}_{N_\eta} \notin U_{N_\xi}$ (because $N_\xi \in N_\eta$, so $\dot{Z} \setminus \bigcap_{x \in \dot{Z} \cap N_\xi} U_x$ is forced to be in $\dot{F} \cap N_\eta$),

(4) $\forall \eta \in E\ \exists h(\eta) < \eta\ \forall \xi \in E \cap \eta\ \xi \geq h(\eta)$ and $t_\xi < t_\eta$ implies that t_η forces that $\dot{x}_{N_\xi} \in \dot{V}_{N_\eta}$.

Replacing E with a stationary subset of E, we get a stronger version of (4) : for all $\xi < \eta$ in E with $t_\xi < t_\eta$, t_η forces that $\dot{x}_{N_\xi} \in \dot{V}_{N_\eta}$.

Letting $\dot{b}$ be an S-name for the generic branch, some condition $s \in S$ forces that $\{\xi \in E : t_{N_\xi} \in \dot{b}\}$ is uncountable, and thus that $\{\dot{x}_{N_\xi} : t_{N_\xi} \in \dot{b}\}$ is a free sequence in $\dot{K}_b$. This shows that some nodes of S (and thus all nodes) force that $\dot{K}$ is not countably tight.

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