Fractured Irrationals

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"Where shall I begin, please your Majesty?" He asked.
"Begin at the beginning," the King said, very gravely, "and go on till you come to the end: then stop." [from Alice in Wonderland]

§1. INTRODUCTION. Here is a simple question.

Problem1. *Guess the next term in the sequence*

A. $2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots.$

B. $\frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \frac{29}{70}, \ldots.$

Don't you hate questions like this? I do. Questions of this sort presume there is a mathematical algorithm which generates the next term. This is not so in the sequence 4,14,34,42,59, which lists the consecutive stops of Manhattan's most famous A-train.

So you have to guess what, if any, algorithm the questioner is using. Of course there are infinitely many correct answers. However, any one familiar with the Fibonacci Sequence would guess $\frac{21}{13}$ for the sequence in Problem1A. Even without naming Fibonacci, it is clear that the numerator of the next term is the sum of the numerator and denominator of the previous term. In Problem1B, the denominator of the term after p/q is p+2q.

Here are two more:

Problem2. *Guess the next term in the sequences*

A. $1, \frac{2}{3}, \frac{3}{4}, \frac{5}{7}, \frac{22}{31}, \frac{28}{39}, \ldots$ (Euler).

B. $\frac{1}{7}, \frac{15}{106}, \frac{16}{113}, \frac{4687}{33102}, \ldots$ (Euclid).

If you succeed in answering those two, then you are very good. In Calculus two, another poorly formulated question about sequences is popular: What is the limit of the sequence? Of course this presumes there is a limit, and you have the tools, in addition to the algorithm of construction, for manipulating the sequence. For example,

Problem3. *What are the limits of these sequences*

A. $\frac{1}{2}, \frac{5}{6}, \frac{13}{12}, \frac{77}{60}, \frac{599}{420}, \ldots$

B. $1, \frac{3}{2}, \frac{20}{12}, \frac{41}{24}, \frac{206}{120}, \ldots$

These really are disguised for example (Problem3A) could be written as

$$\frac{1}{2},\ \frac{1}{2}+\frac{1}{3},\ \frac{1}{2}+\frac{1}{3}+\frac{1}{4},\ \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ldots,$$

the first five terms of the divergent harmonic series. (Problem3B) could be written as

$$1, 1+\frac{1}{2}, 1+\frac{1}{2!}+\frac{1}{3!}, 1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}, 1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!},\ldots,$$

the second through the sixth terms of the Taylor expansion of e^x evaluated at x=1. So presuming we know that algorithm, the sequence converges to e. Note for e we could have also used the sequence

$$1, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, \left(\frac{6}{5}\right)^5, \ldots.$$

When the ancient Greek mathematician Euclid was thinking about π=3.14159265358979323846264338327950288419716939937510..., the expression of fractional quantities as the sum of inverses of integers (such as we did in (Problem3B)) was already well established for two thousand years in Egypt. Euclid's algorithm proves useful to state Tsu Ch'ung Chi's ~500 AD approximation of π (the best for 900 years): $3+\cfrac{1}{7+\cfrac{1}{16}} = 3.1415929\ldots$. You might recall the familiar $3+\frac{1}{7}=3.142857\ldots$.

So it is only necessary to remember the triple 3,7,16 in order to obtain a six-place accuracy for π.. Here is a better approximation

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292}}}} = \frac{103993}{33102} = 3.1415926530$$

The great Indian mathematician, Ramanujan, discovered an algorithm for π,

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2\ddots}}}}}$$

.

An object of the form $q_0+\cfrac{p_1}{q_1+\cfrac{p_2}{q_2+\cfrac{p_3}{q_3+\ddots}}}$ where the p_i and q_i are integers is called (first by John Wallis in 1695) a ***continued fraction*** (or here, ***CF***).

§2. STANDARD CONTINUED FRACTIONS. In these notes we only consider standard continued fractions.

A ***standard continued fraction*** ***(CF)*** has each $p_i=1$. and we use $q_0;<q_1,q_2,q_3,...>$ to denote $q_0+\cfrac{1}{q_1+\cfrac{1}{q_2+\cfrac{1}{q_3+\ddots}}}$. So Tsu Ch'ung Chi's approximation is $3;\langle 7,16\rangle$. For a value of π accurate to 23 decimal places,.try

$$3;\langle 7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,2,2,1,84\rangle$$

When $q_0=0$ we write $<q_1,q_2,q_3,...>$. We adopt of a similar notation for finite continued fractions. When a finite number of consecutive terms of the sequence is constant repeated infinitely often we write a bar over it. So

$$1;\langle 1,1,1,1,1,...\rangle=1;\langle\overline{1}\rangle \text{ and } 2;\langle 1,5,2,3,2,3,2,3,...\rangle=2;\langle 1,5,\overline{2,3,}\rangle.$$

Consider partial CFs of the sequence $1;\langle\overline{1}\rangle$. $1;<1>=1+\frac{1}{1}=2$, $1;<1,1>=1+\cfrac{1}{1+\cfrac{1}{1}}=\frac{3}{2}$, $1;<1,1,1>=\frac{5}{3}$, so the sequence Problem1A might be written $1;<1>$, $1;<1,1>$, $1;<1,1,1>$, $1;<1,1,1,1>$, $1;<1,1,1,1>$. Notice that the infinite version x $=1;<\bar{1}>$ satisfies $x=1+\frac{1}{x}$ or $x^2-x-1=0$. Therefore, $x=\frac{\sqrt{5}+1}{2}$, the **Golden Mean** (ratio) Φ, is the limit of the sequence (Problem 1A).

Problem4. Show that the sequence (Problem 1B) are partial CFs of $\langle\overline{2}\rangle$ and $\sqrt{2}=1;\langle\overline{2}\rangle$.

Problem5. Show that for each n, $\langle\overline{n}\rangle=\frac{n+\sqrt{n^2+4}}{2}$.

Problem6. Determine the real number equal to $1;\langle\overline{1,2}\rangle=1;\langle 1,2,1,2,1,2,..\rangle$.

Given a positive real x, there is an algorithm for determining the CF representing x, which proceeds as follows:

Let [x] represent the *floor function* (the largest integer at most x. Then $p_0=[x]$. Let $y=\frac{1}{x-p_0}$. Then $x=p_0+\frac{1}{y}$ and $y>1$. Let $p_1=[y]$ and $z=\frac{1}{y-p_1}$. Note $x=p_0+\cfrac{1}{p_1+\cfrac{1}{z}}$. Continue this process.

Problem7. Determine a CF for the 4,000 year old Egyptian approximation for π , $4\times\left(\frac{8}{9}\right)^2$.

THEOREM A. The process stops after only finitely many steps iff x is rational.; i.e., x is rational iff it has a finite CF.

This is certainly true about any integer, n. Suppose

* it is true about any rational of the form $\frac{n}{m}$ where n and m are integers and n<m<q.

Given a rational $\frac{p}{q}$, with p<q, $\frac{p}{q}=\frac{1}{\frac{q}{p}}$ now since p<q, there are, by the Euclidean algorithm, integers k and r<q such that $\frac{q}{p}=k+\frac{r}{p}$. Since p<m, * shows that $\frac{r}{p}$ can be written as a CF where the process stops.

Example. Computing the CF of the rational, $\frac{355}{113}$. $\frac{355}{113}=3+\frac{16}{113}=3+\frac{1}{\frac{113}{16}}$.

$\frac{113}{16}=7+\frac{1}{16}$ so $\frac{355}{113}=3+\frac{1}{7+\frac{1}{16}}=3;\langle 7,16\rangle$.

Problem7. Find a CF for the number of inches 100/2.54 in a meter and compare with its decimal representation to the point it repeats.

THEOREM B. Each solution to a quadratic equation is either an integer, or has a CF which is eventually repeating.

Computing the CF of $x=\sqrt{n}$, is very similar to the algorithm for rationals. We will produce an eventually repeating CF for x:

First compute $[x]$. Then x=[x]+x $=[x]+\frac{1}{\frac{1}{x-[x]}}$. In order to determine the floor, rationalize the denominator, and work with the new term in the same fashion. Continue this process and after a while you end with x again.

Example. Compute a continued fraction for $\sqrt{8}$. $\left[\sqrt{8}\right]=2$ So $\sqrt{8}=2+\frac{1}{\frac{1}{\sqrt{8}-2}}$.

Rationalize: $\frac{1}{\sqrt{8}-2}=\frac{\sqrt{8}+2}{4}$. We start the process again: As $\left[\frac{\sqrt{8}+2}{4}\right]=1$, we have

$\sqrt{8} = 2 + \cfrac{1}{1+\left(\frac{\sqrt{8}+2}{4}-1\right)}$. Rationalize: $\frac{\sqrt{8}+2}{4} - 1 = \cfrac{1}{\cfrac{1}{\frac{\sqrt{8}+2}{4}-1}} = \frac{1}{2+\sqrt{8}}$. Thus,

$\sqrt{8} = 2 + \cfrac{1}{1+\cfrac{1}{2+\sqrt{8}}}$ or $\sqrt{8} = 2 + \cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{2+\sqrt{8}}}}}$. Clearly. $\sqrt{8} = 2;\langle\overline{1,4}\rangle$.

Problem 8A. Show $\sqrt{17} = 4;\langle\overline{1,1,1,1,6}\rangle$.

B. Find the CF for the 1,500 year old Hindu approximation for π , $\sqrt{10}$.

C. Find a CF for $\sqrt{13}$.

THEOREM C. Each irrational has exactly one CF representative, and each infinite CF converges.

Just suppose x is an irrational for which we have begun construction as a CF $q_1;\langle q_2, q_3, q_4, ...\rangle$. If we consider only the finite CFs which end at 2n steps $q_1;\langle q_2, q_3, q_4, ..., q_{2n}\rangle$, then we get a decreasing sequence. If we consider only the finite CFs which end at 2n+1 steps $q_1;\langle q_2, q_3, q_4, ..., q_{2n+1}\rangle$, then we get an increasing sequence, and the difference between the last term of the two sequences is at most 1/n (there are better bounds). As 1/n goes to 0, the infinite CF is well defined an equal to x. (The study of how the partial CFs converge to x is extremely interesting and can be found both in books referenced at the end of this note as well as web pages referenced.

L. Euler discovered $e - 1 = 1;\langle 1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,...\rangle$. Note the sequence in (Q2A) are taken from the above Euler formula for e-2. Some more CFs:

$\sqrt{e} - 1 = \langle 1,1,1,\ 5,1,1,\ 9,1,1,\ 13,1,1,\ 17,1,1,\ ...\rangle$,

$\frac{e-1}{e+1} = \langle\ 2,\ 6,\ 10,\ 14,\ ...\rangle$.

$e^{\frac{1}{n}} = 1;\langle n-1.3n-1, 5n-1, 7n-1, ...\rangle$

§3. ORDER. An irrational can be represented by just one CF; however, this is untrue for rationals, $\frac{355}{113} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = 3;\langle 7,15,1\rangle = 3;\langle 7,16\rangle$.

We thus adopt **three CONVENTIONS** for sequences associated with CFs.:

CONVENTION 1. 0 can only occur as the first term. So $1;\langle 1,0,2\rangle$ is not allowed.

CONVENTION 2. At the end of a finite CF we add a terminating term $\frac{1}{\infty}$ where ∞>n. So $1 = 1;\langle\infty\rangle$ and $\frac{19}{5} = 3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\infty}}} = 3;\langle 1,4,\infty\rangle$.

CONVENTION 3. In the expression of a finite CF, if the last non-∞ term is a 1 then we add 1 to the previous step. Thus, for 1 we write $1;\langle\infty\rangle$ not $0;\langle 1,\infty\rangle$, and $\frac{3}{4} = \langle 1,4,\infty\rangle$ not $\langle 1,3,1,\infty\rangle$.

The reason for the conventions is we would like to order both the finite and infinite sequences/CFs in one line: Suppose σ and τ are different sequences, determine the first integer n such that the n'th, σ(n), term of σ differs from the n'th term, τ(n), of τ. If n is odd and σ(n)<τ(n), or if n is even and τ(n)<σ(n), then we will say σ<*τ.

EXAMPLE. $\frac{9}{7} < \frac{11}{8} < \frac{7}{5} < \sqrt{2} < \frac{3}{2}$. According to the above order their representatives satisfy the same order $1;\langle 3,2,\infty\rangle <^* 1;\langle 2,1,1,2,\infty\rangle <^* 1;\langle 2,2,\infty\rangle <^* 1;\langle\bar{2}\rangle <^* 1;\langle 2,\infty\rangle$. We do not allow $\frac{3}{2} = 1;\langle 1,1,\infty\rangle$ because $1;\langle 1,1,\infty\rangle < 1;\langle 1,3,2,\infty\rangle = \frac{9}{7}$.

THEOREM D. The two orders, <* on CFs, and < on their evaluations, agree.

When we speak of intervals of CFs we can use either <* or the usual <.

Problem9. Without using decimals, determine which is bigger under <*, ***π*** or the CF $3;\langle 7,1,15,1,293,\infty\rangle$.

Now we restrict our attention to the set **P** of all irrational numbers identified with the set of infinite CFs. We identify the orders < and <*.

An ***open interval*** between two CFs a and b has the usual definition:

$$(a,b) = \{x\in \mathbf{P} : a < x \text{ and } x < b\}.$$

A set which is the union of open intervals is called an ***open set***. If **P**\C is open, then C is said to be ***closed***.

The set of reals has only one non-empty simultaneously closed and open set - the entire set. Thus, the following has no analog for all the set of real numbers. Given a rational number p=p(1);<p(1),...,p(n)>, define $G_p = \{x\in \mathbf{P}$: the CF for x begins with p$\}$.

EXAMPLE. For p = 5;<1,2>, the set G_p is both open and closed in **P**: Given any $y\in \mathbf{P}$, define $a,b\in \mathbf{P}$, as follows: Let $a = y(1);\left\langle y(2), y(3), y(4)+1, \overline{1}\right\rangle$ and $b = y(1);\left\langle y(2), y(3), y(4), y(5)+1, \overline{1}\right\rangle$. Then a<y<b. Now suppose $y\in G_p$ and $t\in \mathbf{P}$ satisfies both $a\le t\le b$. Then 5=y(1)=a(1)=b(1)=t(1), 1=y(2)=a(2)=t(2), and 2=y(3)=a(3)=t(3). So $t\in G_p$. Thus, $y\in G_p$ implies $(a,b)\subseteq G_p$. Therefore, G is the union of open intervals, and must be open. Now suppose $y\notin G_p$,. Then one of the following must be true: y(1)≠3, y(2)≠1, or y(3)≠2. But this must be true for any $t\in (a,b)$. Therefore, $\mathbf{P}/G_p$ is open, and G_p is closed.

Sets which are both open and closed, are called ***clopen***.

Problem10. Suppose G is the set of $x\in \mathbf{P}$ such that in the real numbers 1<x<2. Prove that G is a clopen set.

Following the last example, we see that each rational determines a clopen set in **P** and:

THEOREM E. If $x\in H$ and H is open, then there is a rational p such that $x\in G_p\subseteq H$.

THEOREM F. A set $C\subseteq \mathbf{P}$ is compact iff it is closed and there is $b\in \mathbf{P}$ such that $\forall\, x\in C$, $\forall\, n$, $x(n)\le b(n)$.

Problem11. Prove that no set G_p is contained in a compact set..

THEOREM G. A set $X\subseteq \mathbf{P}$ is topologically identical to **P** iff X is closed and no open set G of **P** has $G\cap X$ contained in a compact set.

§4. SOME DYNAMICS. When we translate the definition of continuous to our world of irrationals represented by continued fractions, it looks like this:

A function $f:\mathbf{P}\to\mathbf{P}$ is said to be ***continuous at*** x provided for each integer m, there is an integer n so large that if $y(1);<y(2),...,y(n)> = x(1);<x(2),...,x(n)>$, then $f(x)(1);<f(x)(2),...,f(x)(m)> = f(y)(1);<f(y)(2),...,f(y)(m)>$. If it is continuous at each point, then we call it ***continuous***.

Many functions of reals in their ordinary definition fail to be continuous here because that send irrational numbers to rationals. But the usual functions, when considered pointwise remain continuous.

Example. Define $f:\mathbf{P}\to\mathbf{P}$ by $f(x)=x(1)^2;<x(2)^2,x(3)^2,x(4)^2,x(5)^2,....>$. f is continuous and $f(\sqrt{2}) = f(1;\langle\bar{2}\rangle) = 1;\langle\bar{4}\rangle = 2\sqrt{5}-1$.

Define a function $\sigma:\mathbf{P}\to\mathbf{P}$ by $\sigma(x)=\sigma(x)=\dfrac{1}{x-[x]}$ or simply $\sigma(x(1);<x(2),x(3),...>) = x(2);<x(3),x(4),...>$. Because of the second equivalence, where coordinates are shifted to the left, σ is called the ***shift*** map.

THEOREM H. The shift map is continuous.
Since shift merely forgets the first coordinate, and moves the other's to the right, given m, choose n>m. Then if the first n terms of x and y agree, then $x(2);<x(3),x(4),...,x(m)> = \sigma(x(1);<x(2),x(3),...,x(m+1),...>) = \sigma(y(1);<y(2),y(3),...,y(m+1),...>) = y(2);<y(3),y(4),...,y(m)>$. Therefore, σ is continuous at each x.

The ***orbit*** of a point x under $f:\mathbf{P}\to\mathbf{P}$ is the set $\{f(x),f^2(x),f^3(x),...\}$, where $f^{n+1}(x) = f(f^n(x))$.
The ***orbit closure*** $OC_f(x)$ is $\{y\in\mathbf{P}$: for each m, there is an such that $y(1);<y(2),...,y(m)> = f^n(x)(1);<f^n(x(2),...,f^n(x)(m)\}$.

Henceforth, the only continuous function we consider is shift σ.

Problem12. Show $y\in OC(x)$ (for the shift σ) iff for any each integer m there is an m-length segment in x such that $y(1);<y(2),...,y(m)> = x(n+1);<x(n+2),...,x(n+m)\}$.

A finite sequence of integers appearing consecutively in $x\in\mathbf{P}$ is called a ***block*** in x.

Example. <1,1> is a block in ***e*** appearing infinitely many times, while <14,2,1,1,2> and <2,2,2,2> are blocks in ***π***.

Problem13.

A. For x=1;<2,3,4,...>, show the orbit of x is closed.

B. Suppose x∈**P** is such that each finite sequence of positive integers appears as a block in x, Prove OC(x)=**P**.

C. Construct x∈**P** satisfying the hypothesis of B.

THEOREM I. Suppose B is a block in x infinitely often, but in x immediately following B are arbitrarily large integers. Then OC(x) is topologically identical to **P**.

Recall ***e*** = 2;<1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,...>. As the block <1,1> appears in ***e*** infinitely often, OC(***e***) is topologically identical to **P**.

Suppose x∈P. x is said to be ***fixed*** provided that σ(x)=x.

x is ***periodic*** provided there is an n such that for each integer k and each r, 0<r<n, x(kn+r)=x(r).

x is ***almost-periodic*** provided that for each m there is an n such that in each block in x of size n, the initial block of size m appears in x..

x is ***recurrent*** provided each initial block occurs at least one more time.

Example. 1. x is fixed iff there is an n such that $x = \dfrac{n+\sqrt{n^2+4}}{2}$.

2. There is an n such that $\sigma^n(x)$ is periodic iff x is a solution to a quadratic equation.

3. We define α∈**P** by the CF in which α(n)=k+1 if k is the largest power of 2 dividing n. Since $1=2^0$, α(1)=0+1=1. Since $2=2^1$, α(2)=1+1=2. Since $3=3*2^0$, α(3)=0+1=1. Since $4=2^2$, α(4)=2+1=3. So α=1;<2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,...> ≈ 1.35855565.... In each 2^{n+2} term block, the initial 2^n block appears. Thus,α is almost-periodic.

4. We define x∈**P** using α. Replace each n in α by a constant block of n and length n. So x = 1;<2,2,1,3,3,3,1,2,2,1,4,4,4,4,1,2,2,1,3,3,3,1,2,2,1,5,5,5,5,5,...>. x is not almost-periodic because it contains different constant blocks of arbitrarily length. However, x is recurrent.

Problem 14. Prove that OC(α) is topologically identical to **P**.

The last example indicates part of the proof of a theorem and, along with theorem I, tells us how to build recurrent x so that OC(x) is topologically identical to **P**.:

THEOREM J. $x \in \mathbf{P}$ is recurrent iff there is a sequence $\langle B_1, B_2, B_3, \ldots \rangle$ of blocks such that x is the CF constructed by replacing $\alpha(n)$ with $B_{\alpha(n)}$.

References

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