A CHARACTERIZATION OF HILBERT SPACES

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Abstract. Let X be a Banach space with dual X^*, and let $J : X \longrightarrow 2^{X^*}$ be the duality mapping defined by $Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|_X^2, \|x^*\|_{X^*} = \|x\|_X\}$. We prove that if X is a function space so that for every positive simple function $x \in X$ there exists a scalar k_x so that $k_x \cdot x \in J(x)$ then X is isometric to a Hilbert space. This result is valid in both real and complex spaces.

1. Introduction

The purpose of this paper is to present a new characterization of Hilbert spaces through the properties of the duality mapping. Recall that when X is a Banach space and X^* is the dual of X, then the duality mapping (usually multivalued) $J : X \longrightarrow 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|_X^2, \|x^*\|_{X^*} = \|x\|_X\}.$$

The duality mapping is a fundamental concept of nonlinear analysis and its properties have many important implications for the geometric structure of X, see e.g. the monograph [2] for a thorough discussion of properties of the duality mapping and their applications.

Hilbert space is another concept of utmost importance in all fields of analysis, and since the XIX-th century a wealth of conditions characterizing Hilbert spaces have been studied, see the excellent monograph [1] which presents around 350 conditions equivalent to the existence of the inner product, and the very extensive [3] for a comprehensive treatment of the theory and applications of inner product spaces. Several conditions equivalent to the existence of the inner product involve the duality mapping, including one of the most commonly used conditions defining Hilbert spaces. Namely, X is a Hilbert space if and only if

(1.1) For all $x \in X$, $Jx = \{x\}$ (or $Jx = \{\overline{x}\}$, when X is a complex function space).

Other well known conditions involving the duality mapping, include

$$J(x+y) \supseteq Jx + Jy, \quad \text{for all } x, y \in X \qquad [1, (6.7")]$$

$$f \in Jx, g \in Jy \Longrightarrow \text{span}(f, g) \subseteq J(\text{span}(x, y)) \qquad (\text{James [4], cf. [1, (12.12)]})$$

New conditions continue to be identified, e.g. recently Kunze [6] proved that a real space X is Hilbert if and only if the duality map J maps line segments to convex sets.

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In the present note we prove that (real or complex) Hilbert spaces are characterized by a significant weakening of (1.1). Namely, if X is a function space so that for every positive simple function $x \in X$ there exists a scalar k_x so that

$$k_x \cdot x \in J(x) \tag{1.2}$$

then X is isometric to a Hilbert space (see Theorem 3.1). This result is valid in both real and complex spaces. For the characterization to hold it is sufficient that (1.2) is satisfied for any dense subset of X (cf. Remark 3.4).

The precursor of this result was used in [5], where a version of Lemma 3.4 for real nonatomic function spaces was implicit in the proof of [5, Theorem 4.3]. The present paper is self-contained and does not depend on any background material from [5].

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2. Preliminaries

We use standard Banach space notations as may be found e.g. in [7] or [8]. Below we recall basic definitions and facts that we use.

Let us suppose that Ω is a Polish space and that μ is a $\sigma-$finite Borel measure on Ω. We use the term Köthe space in the sense of [7, p. 28]. Thus a *Köthe function space* X on (Ω, μ) is a Banach space of (equivalence classes of) locally integrable Borel functions f on Ω such that:

(1) If $|f| \leq |g|$ a.e. and $g \in X$ then $f \in X$ with $\|f\|_X \leq \|g\|_X$.

(2) If A is a Borel set of finite measure then the characteristic function χ_A of set A belongs to X.

We will also assume that the following normalization condition is always satisfied:

(3) If A is a Borel set with $\mu(A) = 1$ then $\|\chi_A\|_X = 1$.

We say that X is *order-continuous* if whenever $f_n \in X$ with $f_n \downarrow 0$ a.e. then $\|f_n\|_X \downarrow 0$.

The *Köthe dual* of X is denoted X'; thus X' is the Köthe space of all g such that $\int |f||g|\, d\mu < \infty$ for every $f \in X$ equipped with the norm $\|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int |f||g|\, d\mu$. Then X' can be regarded as a closed subspace of the dual X^* of X. If X is order-continuous then $X' = X^*$.

We will be most interested in the situation when (Ω, μ) is one of the following three spaces:

(i) $\Omega =$ positive integers or a finite subset of positive integers, with the counting measure,

(ii) $\Omega = [0, 1]$ with the usual Lebesgue measure,

(iii) $\Omega = [0, \infty)$ with the usual Lebesgue measure.

It follows (cf. [7, Theorem 1.b.14 and pages 118-119] and [8, Theorem 2.7.8]) that in the above three cases $(i) - (iii)$ we have:

(i) when $\Omega = \{1, 2, \dots\}$ or $\Omega = \{1, 2, \dots, n\}$ then, as sets,

$$\ell_1 \subset X \subset \ell_\infty$$

where the dimension of ℓ_1 and ℓ_∞ equals to the cardinality of Ω, and the inclusion maps are of norm one, i.e. if $f \in \ell_1$ then $\|f\|_X \le \|f\|_1$ and if $f \in X$ then $\|f\|_\infty \le \|f\|_X$;

(ii) if $\Omega = [0, 1]$ then, as sets,

$$L_\infty(0, 1) \subset X \subset L_1(0, 1)$$

and the inclusion maps are of norm one, i.e. if $f \in L_\infty(0, 1)$ then $\|f\|_X \le \|f\|_1$ and if $f \in X$ then $\|f\|_\infty \le \|f\|_X$;

(iii) if $\Omega = [0, \infty)$ then, as sets,

$$L_\infty(0, \infty) \cap L_1(0, \infty) \subset X \subset L_1(0, \infty) + L_\infty(0, \infty)$$

and the inclusion maps are of norm one with respect to the natural norms in these spaces, i.e. if $f \in L_\infty(0, \infty) \cap L_1(0, \infty)$ then $\|f\|_X \le \max(\|f\|_1, \|f\|_\infty)$ and if $f \in X$ then $\|f\|_{L_1 + L_\infty} \le \|f\|_X$. Here $L_1(0, \infty) + L_\infty(0, \infty)$ is the space of all functions on $[0, \infty)$ which can be written as $g + h$ with $g \in L_1(0, \infty)$ and $h \in L_\infty(0, \infty)$ and the norm is defined by the formula

$$\|f\|_{L_1 + L_\infty} \stackrel{\text{def}}{=} \inf\{\|g\|_1 + \|h\|_\infty \ : \ f = g + h\} = \sup\{\int_A |f| d\mu : \mu(A) = 1\}.$$

The Köthe dual of $L_1(0, \infty) + L_\infty(0, \infty)$ is order isometric to $L_\infty(0, \infty) \cap L_1(0, \infty)$ endowed with the norm $\max(\|f\|_1, \|f\|_\infty)$ (cf. [7, Proposition 2.a.2]).

Note that in the above, the case (iii) has the most general formulation which we will use in our arguments below. We will write

$$L_\infty(\Omega) \cap L_1(\Omega) \subset X \subset L_1(\Omega) + L_\infty(\Omega)$$

to mean that the appropriate case of (i), (ii), (iii) holds.

3. Main result

Our main result is the following:

Theorem 3.1. *Let X be an order-continuous Köthe function space on (Ω, μ) over $\mathbb{C}$ or $\mathbb{R}$. Suppose that for every positive simple function $g \in X$ there exists $k_g \in \mathbb{C}$ so that*

$$k_g \cdot g \in J(g).$$

Then X is isometric to the Hilbert space on (Ω, μ).

We will prove Theorem 3.1 through the following two lemmas.

Lemma 3.2. *Let X be an order-continuous Köthe function space on (Ω, μ) over $\mathbb{R}$ or $\mathbb{C}$. Suppose that for every positive simple function $x \in X$ there exists a constant $k_x \in \mathbb{C}$ so that*

$$k_x \cdot x \in J(x) \tag{3.1}$$

Then for every $g \in X$, there exists $k_g \in \mathbb{C}$ so that $k_g \cdot \overline{g} \in J(g)$.

Lemma 3.3. *Let X be an order-continuous Köthe function space on (Ω, μ) over $\mathbb{C}$ or $\mathbb{R}$. Suppose that for every function $g \in X$ there exists $k_g \in \mathbb{C}$ so that*

$$k_g \cdot \overline{g} \in J(g).$$

Then X is isometric to the Hilbert space on (Ω, μ).

It is clear that Theorem 3.1 holds once these lemmas are established.

Proof of Lemma 3.2. Let x be any simple function in X. Then $|x|$ is a positive simple function and by (3.1) we have

$$\|x\|^2 = \||x|\|^2 = \int k_{|x|}|x|^2 d\mu = \int k_{|x|} x\overline{x} d\mu.$$

Hence $k_{|x|} \in \mathbb{R}$ and

$$k_{|x|}\overline{x} \in J(x) \tag{3.2}$$

for all simple functions x in X.

Since simple functions are norm dense in X, for any $g \in X$, $g \neq 0$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of simple functions so that for each $n \in \mathbb{N}$,

$$\|\cdot\|_X - \lim_{n\to\infty} x_n = g. \tag{3.3}$$

By (3.2) for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{R}$ so that

$$k_n\overline{x}_n \in J(x_n).$$

Clearly, for all $n \in \mathbb{N}$, $k_n > 0$. Further

$$\|x_n\|_X = \|k_n\overline{x}_n\|_{X^*} = k_n\|\overline{x}_n\|_{X^*}. \tag{3.4}$$

We also have, for all x in X

$$\|\overline{x}_n\|_{X^*} = \|x_n\|_{X^*} \geq \|x_n\|_{L_1+L_\infty},$$

and thus by (3.3), we get

$$\|\cdot\|_{L_1+L_\infty} - \lim_{n\to\infty} \overline{x}_n = \overline{g}. \tag{3.5}$$

Therefore there exists $N \in \mathbb{N}$ so that for all $n \geq N$:

$$\|x_n\|_{L_1+L_\infty} \geq \frac{1}{2}\|g\|_{L_1+L_\infty},$$
$$\|x_n\|_X \leq 2\|g\|_X.$$

Hence, for all $n \geq N$

$$0 \leq k_n = \frac{\|x_n\|_X}{\|x_n\|_{X^*}} \leq \frac{4\|g\|_X}{\|g\|_{L_1+L_\infty}}.$$

Thus sequence $(k_n)_{n\in\mathbb{N}}$ is bounded. By passing to a subsequence, if necessary, we will assume without loss of generality that the sequence $(k_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is convergent. Denote

$$k_0 = \lim_{n\to\infty} k_n \in \left[0, \frac{4\|g\|_X}{\|g\|_{L_1+L_\infty}}\right]. \tag{3.6}$$

Next we consider the sequence $(k_n\overline{x}_n)_{n\in\mathbb{N}} \subset X^*$. Since for all $n \in \mathbb{N}$, $k_n\overline{x}_n \in J(x_n)$ and the sequence $(\|x_n\|_X)_{n\in\mathbb{N}}$ is bounded we conclude that $(k_n\overline{x}_n)_{n\in\mathbb{N}}$ is norm-bounded in X^*. Since the unit ball of X^* is weak*-compact we conclude that there is a subnet $(k_\alpha\overline{x}_\alpha)_{\alpha\in\Lambda}$ of $(k_n\overline{x}_n)_{n\in\mathbb{N}}$ which is weak*-convergent, say

$$w^* - \lim_{\alpha\in\Lambda}(k_\alpha\overline{x}_\alpha) = \varphi \in X^*. \tag{3.7}$$

Since the inclusion $X^* \hookrightarrow L_1 + L_\infty$ is $\sigma(X^*, X) - \sigma(L_1 + L_\infty, L_1 \cap L_\infty)$ continuous, (3.7) implies that $(k_\alpha\overline{x}_\alpha)_{\alpha\in\Lambda}$ converges to φ in $\sigma(L_1+L_\infty, L_1\cap L_\infty)-$topology. On the other hand, (3.5) and (3.6) imply that $(k_\alpha\overline{x}_\alpha)_{\alpha\in\Lambda}$ converges to $k_0\overline{g}$ in $(L_1 + L_\infty)-$norm. Thus

$$\varphi = k_0\overline{g}$$

as elements of $L_1 + L_\infty$.

To finish the proof we note that it follows from (3.7) and [2, Lemma II.1.7] that

$$\varphi \in J(g). \tag{3.8}$$

For the convenience of the reader we repeat the short argument here. Indeed for all $x \in X$

$$\langle \varphi, x\rangle = \lim_{n\to\infty} \langle k_n\overline{x}_n, x\rangle.$$

Thus

$$\|\varphi\|_{X^*} \leq \lim_{n\to\infty} \|k_n\overline{x}_n\|_{X^*} = \lim_{n\to\infty} \|x_n\|_X = \|g\|_X.$$

Further

$$\begin{aligned}
\langle \varphi, g\rangle &= \lim_{n\to\infty} \langle k_n\overline{x}_n, g\rangle \\
&= \lim_{n\to\infty} (\langle k_n\overline{x}_n, x_n\rangle + \langle k_n\overline{x}_n, g - x_n\rangle) \\
&= \lim_{n\to\infty} (\|x_n\|_X^2 + \langle k_n\overline{x}_n, g - x_n\rangle) \\
&= \|g\|^2,
\end{aligned}$$

since for n large enough:

$$\begin{aligned}
|\langle k_n\overline{x}_n, g - x_n\rangle| &\leq \|k_n\overline{x}_n\|_{X^*}\ \|g - x_n\|_X \\
&= \|x_n\|_X\ \|g - x_n\|_X \\
&\leq 2\|g\|_X\ \|g - x_n\|_X,
\end{aligned}$$

and therefore $\lim_{n\to\infty}\langle k_n\overline{x}_n, g-x_n\rangle = 0$, which ends the proof of (3.8) and of the lemma. □

Proof of Lemma 3.4. It is enough to show that for every $g \in X$, $k_g = 1$, since then for every $g \in X$, $\overline{g} \in J(g)$ and therefore

$$\|g\|_X^2 = \int_\Omega g \cdot \overline{g}\, d\mu = \int_\Omega |g|^2\, d\mu,$$

i.e. X is isometric to $L_2(\Omega, \mu)$.

Let $A \subseteq \Omega$ be a set with $\mu(A) = 1$. Then, by the normalization assumption, $\|\chi_A\|_X = 1$ and

$$\|\chi_A\|_X^2 = 1 = \int_A 1^2\, d\mu,$$

so $k_{\chi_A} = 1$. Thus it is enough to show that for every $f, g \in X$ we have

$$k_f = k_g. \tag{3.9}$$

We will show that (3.9) holds in two steps. First we will show that (3.9) is valid for functions $f, g \in X$ such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ (here, and in the following all set relations are considered modulo sets of measure zero), and then we will deduce the general case. We start by noting that, clearly, for any $f \in X$ and any $a \in \mathbb{C} \setminus \{0\}$,

$$k_{(af)} = k_f. \tag{3.10}$$

Let $f, g \in X$ be such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ and $\|f\|_X = \|g\|_X = 1$. Then

$$1 = \|f\|_X^2 = \int k_f f\overline{f}\, d\mu = k_f \int |f|^2\, d\mu \tag{3.11}$$

$$1 = \|g\|_X^2 = \int k_g g\overline{g}\, d\mu = k_g \int |g|^2\, d\mu \tag{3.12}$$

$$\int |f| \cdot |g|\, d\mu = 0. \tag{3.13}$$

It follows from (3.11) and (3.12) that k_f, k_g are positive real numbers. For any $\alpha \in [0, 2\pi]$ set

$$F(\alpha) = \sqrt{k_f}|f| \cos\alpha + \sqrt{k_g}|g| \sin\alpha,$$
$$H(\alpha) = \|F(\alpha)\|_X.$$

We note that H is a real valued Lipschitz function on $[0, 2\pi]$ with Lipschitz constant not exceeding $\sqrt{k_f} + \sqrt{k_g}$. We will show that $H'(\alpha) = 0$ a.e. and deduce that H is constant.

Let us suppose that θ is a point of differentiability of H. Then $H(\alpha) - \langle F(\alpha), k_{F(\theta)}F(\theta)\rangle$ has a minimum at $\alpha = \theta$ and so we can deduce that $H'(\theta) = \langle F'(\theta), k_{F(\theta)}F(\theta)\rangle$. We have, by (3.11), (3.12), (3.13),

$$\begin{aligned}\langle F'(\theta), k_{F(\theta)}F(\theta)\rangle &= \int k_{F(\theta)}F(\theta)F'(\theta)\ d\mu \\ &= k_{F(\theta)}\int \left(\sqrt{k_f}|f|\cos(\theta)+\sqrt{k_g}|g|\sin(\theta)\right)\left(-\sqrt{k_f}|f|\sin(\theta)+\sqrt{k_g}|g|\cos(\theta)\right)\ d\mu \\ &= k_{F(\theta)}\left[\cos(2\theta)\int \sqrt{k_f}\sqrt{k_g}|f|\cdot|g|\ d\mu - \frac{1}{2}\sin(2\theta)\int (k_f|f|^2-k_g|g|^2)\ d\mu\right] \\ &= 0.\end{aligned}$$

Thus H is constant as promised. Notice that:

$$H(0) = \|\sqrt{k_f}f\|_X = \sqrt{k_f},$$
$$H(\frac{\pi}{2}) = \|\sqrt{k_g}g\|_X = \sqrt{k_g}.$$

Thus $k_f = k_g$. Moreover

$$H(\frac{\pi}{4}) = \|\frac{\sqrt{k_f}}{\sqrt{2}}(f+g)\|_X = \sqrt{k_f}.$$

Hence $\|f+g\|_X = \sqrt{2}$ and since f and g are disjoint we get

$$\begin{aligned}2 = \|f+g\|_X^2 &= \int k_{f+g}|f+g|^2\ d\mu \\ &= \int k_{f+g}|f|^2 + k_{f+g}|g|^2\ d\mu \\ &= \frac{k_{f+g}}{k_f}\int k_f|f|^2\ d\mu + \frac{k_{f+g}}{k_g}\int k_g|g|^2\ d\mu \\ &= \frac{k_{f+g}}{k_f} + \frac{k_{f+g}}{k_g} = \frac{2k_{f+g}}{k_f}.\end{aligned}$$

Thus whenever f, g are disjoint

$$k_{f+g} = k_f = k_g. \tag{3.14}$$

Now let x, y be any nonzero functions in X. Then there exists a set $A \subset \Omega$ so that functions $x_1 = x\chi_A,\ x_2 = x\chi_{A^c},\ y_1 = y\chi_A,\ y_2 = y\chi_{A^c}$ are all nonzero. By (3.14) we get

$$k_x = k_{(x_1+x_2)} = k_{x_1} = k_{y_2} = k_{y_1} = k_{(y_1+y_2)} = k_y.$$

which proves (3.9) and ends the proof of the lemma. □

Remark 3.4. It is clear from the proof that Lemma 3.2 remains valid under the assumption that (3.1) holds for every positive function from the set $\mathcal{A}$ with the property that

$$\tilde{\mathcal{A}} = \{h\cdot x : x \in \mathcal{A}, |h| = 1 a.e.\}$$

is dense in X. Therefore also Theorem 3.1 remains valid under this weaker assumption.

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