

ON THE STRUCTURE OF LEVEL SETS OF UNIFORM AND LIPSCHITZ QUOTIENT MAPPINGS FROM \mathbb{R}^n TO \mathbb{R}

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ABSTRACT. We study two questions posed by Johnson, Lindenstrauss, Preiss, and Schechtman, concerning the structure of level sets of uniform and Lipschitz quotient mappings from $\mathbb{R}^n \rightarrow \mathbb{R}$. We show that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $n \geq 2$, is a uniform quotient mapping then for every $t \in \mathbb{R}$, $f^{-1}(t)$ has a bounded number of components, each component of $f^{-1}(t)$ separates \mathbb{R}^n and the upper bound of the number of components depends only on n and the moduli of co-uniform and uniform continuity of f .

Next we obtain a characterization of the form of any closed, hereditarily locally connected, locally compact, connected set with no end points and containing no simple closed curve, and we apply it to describe the structure of level sets of co-Lipschitz uniformly continuous mappings $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We prove that all level sets of any co-Lipschitz uniformly continuous mapping from \mathbb{R}^2 to \mathbb{R} are locally connected, and we show that for every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r \rightarrow 0} \Omega(r) = 0$, there exists a natural number $M = M(c, \Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω , there exists a natural number $n(f) \leq M$ and a finite set $T_f \subset \mathbb{R}$ with $\text{card}(T_f) \leq n(f) - 1$ so that for all $t \in \mathbb{R} \setminus T_f$, $f^{-1}(t)$ has exactly $n(f)$ components, $\mathbb{R}^2 \setminus f^{-1}(t)$ has exactly $n(f) + 1$ components and each component of $f^{-1}(t)$ is homeomorphic with the real line and separates the plane into exactly 2 components. The number and form of components of $f^{-1}(s)$ for $s \in T_f$ are also described – they have a finite graph structure.

We give an example of a uniform quotient map from \mathbb{R}^2 to \mathbb{R} which has non-locally connected level sets.

1. INTRODUCTION

Let X, Y be metric spaces. A mapping $f : X \rightarrow Y$ is said to be a *uniform quotient mapping* if there exist functions $\omega, \Omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\omega(r) > 0$ for all $r > 0$ and $\lim_{r \rightarrow 0} \Omega(r) = 0$ so that for all $x \in X$ and all $r > 0$:

$$(1.1) \quad B(f(x), \omega(r)) \subset f(B(x, r)) \subset B(f(x), \Omega(r)),$$

where $B(x, r)$ denotes the open ball with center x and radius r .

Notice that the right hand inclusion means that f is uniformly continuous. The mapping f is called *co-uniformly continuous* if the left hand inclusion in (1.1) is satisfied. There is no restriction in assuming that the functions ω and Ω are continuous and increasing. They are called *moduli of co-uniform and uniform continuity of f* , respectively. If the functions ω and Ω are linear, i.e. if there exist constants $c, L > 0$ so that for all $x \in X$ and all $r > 0$:

$$(1.2) \quad B(f(x), cr) \subset f(B(x, r)) \subset B(f(x), Lr),$$

then f is called a *Lipschitz quotient mapping*. Clearly the right hand inclusion in (1.2) means that f is a Lipschitz mapping. If f satisfies the left hand inclusion of (1.2), f is called a

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co-Lipschitz mapping. Constants c and L are called *co-Lipschitz and Lipschitz constants of f* , respectively. The study of uniform and Lipschitz quotient mappings was initiated in [1], see also [3] for the comprehensive introduction of the subject. The structure of Lipschitz and uniform quotient mappings $f : X \rightarrow Y$, when X and Y are finite dimensional was studied by Johnson, Lindenstrauss, Preiss and Schechtman in [8]. They obtained most complete results for the case of $X = Y = \mathbb{R}^2$. For $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ they proved, in particular, that if f is a uniformly continuous and co-Lipschitz, e.g. if f is a Lipschitz quotient mapping, then for every $t \in \mathbb{R}^2$, $f^{-1}(t)$ is a finite set of points in \mathbb{R}^2 and $f = P \circ h$ where h is a homeomorphism of the plane and P is a complex polynomial (see also Remark 5.2 below). The question whether level sets of $f^{-1}(t)$ of a Lipschitz quotient map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are discrete, is open for all $n > 2$.

In [8], the authors also study the structure of level sets $f^{-1}(t)$ of uniform and Lipschitz maps $f : \mathbb{R}^n \rightarrow \mathbb{R}$. They showed, among others, the following results:

Theorem 1.1. [8, Proposition 5.1] *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a uniform quotient mapping satisfying (1.1). Then for each $t \in \mathbb{R}$ the number of components of $\mathbb{R}^n \setminus f^{-1}(t)$ is finite and bounded by a function of n , $\omega(\cdot)$ and $\Omega(\cdot)$ only.*

Theorem 1.2. [8, Proposition 5.4] *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz quotient mapping. Then, for each $t \in \mathbb{R}$, each component of $f^{-1}(t)$ is unbounded and separates the plane.*

Theorem 1.3. [8, Corollary 5.5] *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz quotient mapping. Then, for each $t \in \mathbb{R}$, $f^{-1}(t)$ has a bounded number of components. The upper bound of the number of components depends only on the Lipschitz and co-Lipschitz constants of f .*

They also asked the following two questions:

- (Q1) Can one weaken the assumption of Lipschitz quotient to uniform quotient mappings in Theorems 1.2 and 1.3?
- (Q2) To what extent is the number of components of $f^{-1}(t)$ or of $\mathbb{R}^2 \setminus f^{-1}(t)$ independent of t ? Are these numbers constant after excluding finitely many values of t ?

Question (Q2) is motivated by the following two examples of Lipschitz quotient mappings from \mathbb{R}^2 to \mathbb{R} . In both cases the mapping f is the ℓ_1 distance from the solid lines multiplied, in each component of the complement of the solid lines, by the sign indicated.

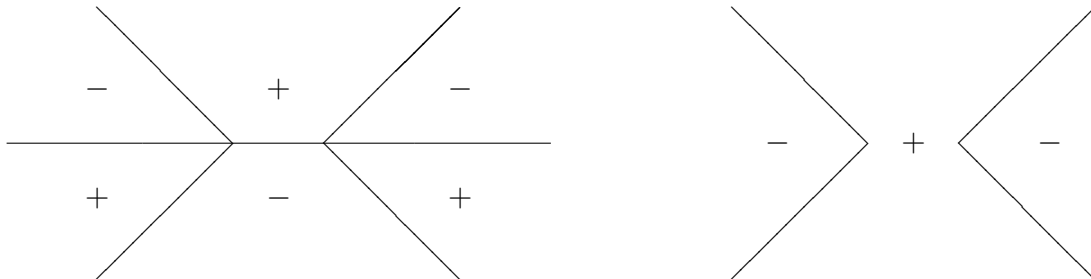


FIGURE 1.1.

Here $f^{-1}(0)$ has one component in the first example and two in the second, and $\mathbb{R}^2 \setminus f^{-1}(0)$ has six and three components, respectively. The authors of [8] note that it is easy to draw examples with an arbitrary finite number of components of $f^{-1}(0)$. Thus question (Q2) is essentially asking whether all Lipschitz quotient maps \mathbb{R}^2 to \mathbb{R} have the form similar to the examples illustrated in Figure 1.1.

This paper is devoted to the study of questions (Q1) and (Q2). We answer both of them affirmatively. First, in Section 2, we obtain generalizations of Theorems 1.3 and 1.2 for uniform quotient mappings from \mathbb{R}^n to \mathbb{R} for any $n \geq 2$ (Theorem 2.4) and Corollary 2.5, respectively). Our results follow from Theorem 1.1 through general topological arguments based on the Phragmen-Brower theorem and the theory of separation in \mathbb{R}^n .

Next we study question (Q2). We obtain not only information about the number of components of $f^{-1}(t)$ and of $\mathbb{R}^2 \setminus f^{-1}(t)$ for Lipschitz quotient maps $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, but we give the full characterization of both the number and the form of each component of any level set $f^{-1}(t)$ of co-Lipschitz uniformly continuous mappings $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ (Theorem 5.1). We show that for every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r \rightarrow 0} \Omega(r) = 0$, there exists a natural number $M = M(c, \Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω , there exists a natural number $n(f) \leq M$ and a finite set $T_f \subset \mathbb{R}$ with $\text{card}(T_f) \leq n(f) - 1$ so that for all $t \in \mathbb{R} \setminus T_f$, $f^{-1}(t)$ has exactly $n(f)$ components, $\mathbb{R}^2 \setminus f^{-1}(t)$ has exactly $n(f) + 1$ components and each component of $f^{-1}(t)$ is homeomorphic with the real line and separates the plane into exactly 2 components. The number and form of components of $f^{-1}(s)$ for $s \in T_f$ is also described – these components have a finite graph structure (for precise formulation see Theorems 5.1, 4.11 and Remark 5.3).

Thus we do confirm that co-Lipschitz uniformly continuous mappings from \mathbb{R}^2 to \mathbb{R} have a form analogous to the examples presented on Figure 1.1. Moreover, we prove that, as on Figure 1.1, no level set $f^{-1}(t)$ can contain parallel lines, but the distance between unbounded components of $f^{-1}(t) \setminus B(0, R)$ has to increase to infinity as R increases to infinity, cf. Figure 1.2 (Proposition 5.8).

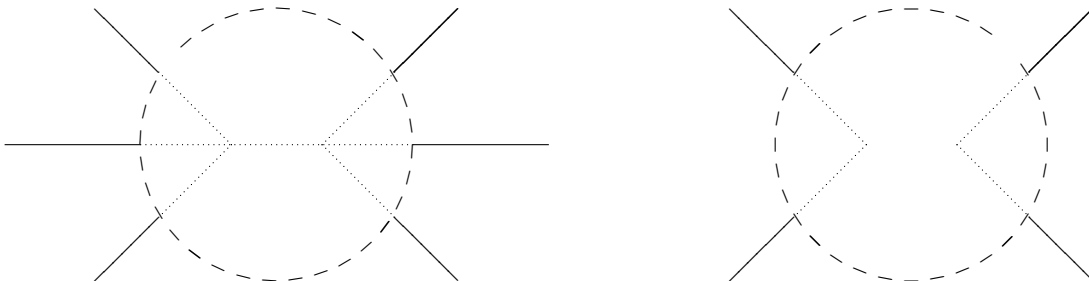


FIGURE 1.2.

Our method of proof of Theorem 5.1 depends on a careful analysis of topological properties of level sets $f^{-1}(t)$, their end points and their structure at infinity. The crucial property that we use in a very essential way is the fact that level sets $f^{-1}(t)$ are locally connected when f is a co-Lipschitz uniformly continuous map from \mathbb{R}^2 to \mathbb{R} (Proposition 3.5).

Our first characterization of the structure of level sets is in fact a characterization of the form of any closed, hereditarily locally connected, locally compact, connected set with no end points and containing no simple closed curve. We present a new self-contained proof of this characterization. Similar characterizations for dendrites and for sets whose every point is a cut point have been previously obtained by Shimrat [18], Stone [20] and Nadler [15] (see Theorem 4.11 and Remarks 4.13, 4.14).

We do not know whether level sets of co-Lipschitz uniformly continuous maps or of Lipschitz quotient maps from \mathbb{R}^n to \mathbb{R} are locally connected when $n > 2$. If one looks for a counter-example, the most natural map to check would be the Lipschitz quotient map

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ constructed by Csörnyei [4], whose level set $f^{-1}(0)$ is very large and complicated. It turns out, however, that for this map and also for its both coordinate maps, which go from \mathbb{R}^3 to \mathbb{R} , all level sets are locally connected.

However we do know that there exist uniform quotient maps from \mathbb{R}^2 to \mathbb{R} with non-locally connected level sets (see Example 6.1).

The local connectedness of level sets $f^{-1}(t)$ of a co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, allows us to use the notion of ends from the algebraic topology (cf. [7], see Definition 4.9) to analyze the behavior of level sets at infinity and consequently to fully describe the topological structure of level sets and their complements (which is achieved in Sections 4 and 5).

Throughout the paper we use standard notation, as may be found in [3, 10, 11, 23].

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2. NUMBER OF COMPONENTS OF LEVEL SETS OF UNIFORM QUOTIENT MAPPINGS FROM \mathbb{R}^n TO \mathbb{R} IS FINITE

As a corollary of Theorem 1.1 using purely topological arguments we will show that when $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a uniform quotient mapping then for each $t \in \mathbb{R}$, the number of components of $f^{-1}(t)$ is finite (Theorem 2.4 below). Our main tool is the following fact:

Theorem 2.1. *Let $B_0, B_1 \subset \mathcal{S}_n$, $n \geq 2$, be two closed sets such that $B_0 \cap B_1 \subseteq \{q\}$ a singlepoint. If none of the sets B_0 or B_1 separates between points p_1 and p_2 then their union $B_0 \cup B_1$ does it neither.*

The above statement combines [11, Theorem 59.II.11 and 61.I.7] specialized to the situation in the present paper. In the case when $n = 2$, Kuratowski refers to this fact as the *first theorem of Janiszewski*, and it's general version is called the *Phragmen-Brouwer theorem*.

Although the subject is closely related to some classical duality theorems, cf. [16, 19, 2], we were unable to find in the literature results that we could directly use in the situation we deal with. We decided to present a proof of the fact we needed, based on some standard arguments concerning separation in \mathbb{R}^n .

We start from two lemmas.

Lemma 2.2. *Let A be an open connected subset of \mathcal{S}_n , so that $\overline{A} \neq \mathcal{S}_n$ and $\text{Bd}(A) = F_1 \cup F_2$ where F_1, F_2 are closed sets with $F_1 \cap F_2 \subseteq \{q\}$ a singlepoint. Let $p_1 \in A$ and $p_2 \notin \overline{A}$. Then exactly one of the sets F_1 or F_2 separates between p_1 and p_2 .*

Proof. By Theorem 2.1 we conclude that at least one of the sets F_1 or F_2 separates p_1 and p_2 . Suppose now that each of F_1 and F_2 separates between p_1 and p_2 . Then there exist components C_1, C_2 of $\mathcal{S}_n \setminus F_1, \mathcal{S}_n \setminus F_2$ respectively so that

$$\begin{aligned} p_1 &\in C_1 \cap C_2, \text{ and thus } A \subset C_1 \cap C_2, \\ p_2 &\notin C_1 \cup C_2. \end{aligned}$$

Then

$$\text{Bd}(C_1 \cup C_2) \subset \text{Bd}(C_1) \cup \text{Bd}(C_2) \subset F_1 \cup F_2.$$

Let $x \in F_1 \setminus \{q\}$. Then for every neighborhood V_x of x we have $V_x \cap A \neq \emptyset$, since $x \in \text{Bd}(A)$. Thus $V_x \cap C_2 \neq \emptyset$ and $x \in \overline{C_2}$. Since $x \notin F_2$ we conclude that $x \in C_2$ and therefore $x \notin \text{Bd}(C_1 \cup C_2)$. Similarly, if $y \in F_2 \setminus \{q\}$ then $y \notin \text{Bd}(C_1 \cup C_2)$. Thus $\text{Bd}(C_1 \cup C_2) \subset \{q\}$ which contradicts the fact that $p_2 \notin C_1 \cup C_2$. \square

Lemma 2.3. *Let A be an open connected subset of \mathcal{S}_n so that $A \neq \mathcal{S}_n$ and $\text{Bd}(A) = F_1 \cup F_2$ where F_1, F_2 are closed sets with $F_1 \cap F_2 \subseteq \{q\}$ a singlepoint. Suppose that $\mathcal{S}_n \setminus F_1$ is connected. Then for every $x \in F_1 \setminus \{q\}$ there exists a neighborhood U_x of x so that $U_x \subset \overline{A}$.*

Proof. Let $x \in F_1 \setminus \{q\}$. Since F_2 is closed, there exists a connected neighborhood U_x of x so that $U_x \cap F_2 = \emptyset$. Since $x \in \text{Bd}(A)$, there exists $y \in U_x$ so that $y \in A$. Suppose that $U_x \setminus \overline{A} \neq \emptyset$ and let $z \in U_x \setminus \overline{A}$. Then $\text{Bd}(A)$ separates between the points y and z . But $\mathcal{S}_n \setminus F_1$ is connected so F_1 does not separate between y and z . Thus by Theorem 2.1 we conclude that F_2 separates between y and z . But this is a contradiction since y, z belong to a connected set U_x which is disjoint with F_2 . \square

With these tools we are ready to prove the main theorem of this section.

Theorem 2.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a uniform quotient map. Then, for any $t \in \mathbb{R}$, a number of connected components of $f^{-1}(t)$ is finite and bounded by a function of n , $\omega(\cdot)$ and $\Omega(\cdot)$ only.*

Proof. We consider \mathbb{R}^n as embedded in its one point compactification \mathcal{S}_n . Denote $K = f^{-1}(t)$. By [8, Lemma 5.2], K is unbounded and therefore the closure of K in \mathcal{S}_n equals $K \cup \{\infty\}$, and the closure in \mathcal{S}_n of every component of K contains $\{\infty\}$. By Theorem 1.1, $\mathbb{R}^n \setminus K$ and therefore also $\mathcal{S}_n \setminus \overline{K}$ has a finite number of components, say

$$\mathcal{S}_n \setminus \overline{K} = \bigcup_{j=1}^m C_j.$$

Here $C_j \subset \mathcal{S}_n \setminus \{\infty\}$, so each C_j can also be considered as a subset of \mathbb{R}^n . Note that C_j cannot be bounded in \mathbb{R}^n , so $\infty \in \text{Bd}(C_j) \subset \mathcal{S}_n$ for all j . Suppose that there exists j , say $j = 1$, so that $\text{Bd}(C_1)$ has m or more connected components in \mathbb{R}^n . Then $\text{Bd}(C_1) \subset \mathbb{R}^n$ can be presented as a sum of m disjoint closed sets in \mathbb{R}^n , which are not necessarily connected. Thus after taking closures in \mathcal{S}_n we see that

$$\text{Bd}(C_1) = F_1 \cup \dots \cup F_m,$$

where $\{F_k\}_{k=1}^m$ are closed sets in \mathcal{S}_n , not necessarily connected, so that $F_k \cap F_l \subseteq \{\infty\}$ for all $k \neq l$.

Let $\{p_j\}_{j=1}^m$ be a collection of points such that $p_j \in C_j$ for $j = 1, \dots, m$. Since for each $j = 2, \dots, m$, $\text{Bd}(C_1)$ separates between p_1 and p_j , by Theorem 2.1, there exists $\sigma(j) \in \{1, \dots, m\}$ so that $F_{\sigma(j)}$ separates between p_1 and p_j . By Lemma 2.2, $\text{Bd}(C_1) \setminus F_{\sigma(j)}$ does not separate between p_1 and p_j , so the choice of $\sigma(j)$ is unique. Thus $\text{card}(\{\sigma(j)\}_{j=2}^m) \leq m - 1$. Hence there exists $j_0 \in \{1, \dots, m\}$ so that F_{j_0} does not separate between p_1 and p_i for all $i = 2, \dots, m$. Thus $\mathcal{S}_n \setminus F_{j_0}$ is connected, and by Lemma 2.3 for every $x \in F_{j_0} \setminus \{\infty\}$ there exists a neighborhood U_x of x so that $U_x \subset \overline{C_1}$. But $f(C_1) \subset (t, \infty)$ or $f(C_1) \subset (-\infty, t)$, thus $f(U_x) \subset [t, \infty)$ or $f(U_x) \subset (-\infty, t]$, which contradicts the fact that $f(U_x) \supset B(f(x), \varepsilon) = (t - \varepsilon, t + \varepsilon)$ for some $\varepsilon > 0$. This contradiction yields that $\text{Bd}(C_1)$ has at most $(m - 1)$ components in \mathbb{R}^n . Similarly, for every $j \in \{1, \dots, m\}$, $\text{Bd}(C_j)$ has at most $(m - 1)$ components and since every component of K contains a component of $\text{Bd}(C_j)$

for at least one $j \in \{1, \dots, m\}$, we conclude that the number of components of K is smaller or equal than $m(m-1)$. \square

Corollary 2.5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a uniform quotient mapping. Then, for each $t \in \mathbb{R}$, each component of $f^{-1}(t)$ separates \mathbb{R}^n .*

This Corollary has word for word the same proof as [8, Proposition 5.4], since by Theorem 2.4, for each $t \in \mathbb{R}$, $f^{-1}(t)$ has a finite number of components.

3. LOCAL CONNECTEDNESS OF LEVEL SETS

In this section we show that all level sets of co-Lipschitz uniformly continuous mappings from \mathbb{R}^2 to \mathbb{R} are hereditarily locally connected. This is a very strong property which will enable us to give a detailed description of the structure of the level sets, see Sections 4 and 5.

As mentioned in the Introduction, we do not know whether there exist co-Lipschitz uniformly continuous maps or Lipschitz quotient maps from \mathbb{R}^3 to \mathbb{R} , or in general from \mathbb{R}^n to \mathbb{R}^k , with non-locally connected level sets. However we do know that there exist uniform quotient maps from \mathbb{R}^2 to \mathbb{R} which have non-locally connected level sets (see Section 6).

We begin by recalling some basic definitions.

Definition 3.1. A topological space S is said to be *locally connected at a point x* if for every open set U containing x there is a connected open set V so that $x \in V \subset U$. The space S is *locally connected* if it is locally connected at each point and S is *hereditarily locally connected* if every subcontinuum of S is locally connected.

We will use the following characterization of hereditary local connectedness:

Theorem 3.2. [23, V.(2.1) and I.(12.2)] *A locally compact connected set S is hereditarily locally connected if and only if S does not contain a continuum of convergence.*

Recall that if a continuum K is a subset of a set M then K is called a *continuum of convergence* of M provided that there exists in M a sequence of mutually exclusive continua K_1, K_2, \dots , no one of which contains a point of K and which converges to K as a limit, i.e. $K \cap \bigcup_{i=1}^{\infty} K_i = \emptyset$ and $\lim[K_i]_i = K$.

Here $\lim[K_i]_i$ denotes the limit of a sequence $[K_i]_i$ which is defined as follows (cf. [23, Section I.7] or [10, Chapter 11, Section 29]): The set of all points x such that every neighborhood of x contains points of infinitely many sets of $[K_i]_i$ is called the *limit superior* of $[K_i]_i$ and is denoted $\limsup[K_i]_i$. The set of all points y such that every neighborhood of y contains points of all but a finite number of the sets $[K_i]_i$ is called the *limit inferior* of $[K_i]_i$ and is denoted $\liminf[K_i]_i$. If $\limsup[K_i]_i = \liminf[K_i]_i$ then we say that the collection $[K_i]_i$ is *convergent* and we write $\lim[K_i]_i = \limsup[K_i]_i = \liminf[K_i]_i$ and we call $\lim[K_i]_i$, the *limit* of $[K_i]_i$.

We will prove that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a co-Lipschitz uniformly continuous mapping then for all $t \in \mathbb{R}$, $f^{-1}(t)$ does not contain a continuum of convergence. For this we will need the following “bottleneck lemma”, whose proof is very similar to the proof of [8, Lemma 5.3].

Lemma 3.3. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map with co-Lipschitz constant 1 and a modulus of uniform continuity Ω . Let K_1, K_2 be disjoint subcontinua of $f^{-1}(0)$ and let $\alpha \in \mathbb{R}_+$. If there exist points $x_1, x_2 \in K_1$, $y_1, y_2 \in K_2$ so that, for $i = 1, 2$,*

$$d(x_i, y_i) \leq \alpha,$$

then

$$d(x_1, x_2) \leq 2\Omega\left(\frac{\alpha}{2}\right) + 4\alpha.$$

For the proof we will need the following basic lemma concerning the lifting of Lipschitz curves which was established in [1].

Lemma 3.4. [1, Lemma 4.4] *Suppose that $f : \mathbb{R}^n \rightarrow X$ is continuous and co-Lipschitz with constant one, $f(x) = y$, and $\xi : [0, \infty) \rightarrow X$ is a curve with Lipschitz constant one, and $\xi(0) = y$. Then there is a curve $\phi : [0, \infty) \rightarrow \mathbb{R}^n$ with Lipschitz constant one such that $\phi(0) = x$ and $f(\phi(t)) = \xi(t)$ for $t \geq 0$.*

Proof of Lemma 3.3. If $d(x_1, x_2) \leq 2\alpha$ then we are done, so assume without loss of generality that $d(x_1, x_2) > 2\alpha$. For $i = 1, 2$, let I_i be the segment connecting x_i and y_i , i.e. $I_i = \{(1-t)x_i + ty_i : t \in [0, 1]\}$. Then $\text{length}(I_i) \leq \alpha$, for $i = 1, 2$, and thus, if $d(x_1, x_2) > 2\alpha$ then $I_1 \cap I_2 = \emptyset$. Set

$$t_i = \sup\{t \in [0, 1] : (1-t)x_i + ty_i \in K_1\}.$$

Since K_1 and I_i are compact and $y_i \notin K_1$ we get that $t_i \in [0, 1)$. Define

$$\bar{x}_i \stackrel{\text{def}}{=} (1-t_i)x_i + t_i y_i \in K_1.$$

Now set

$$s_i = \inf\{t \in [t_i, 1] : (1-t)x_i + ty_i \in K_2\}.$$

Similarly as above, since K_2 is compact and $\bar{x}_i \notin K_2$, we get that $s_i \in (t_i, 1]$. Define

$$\bar{y}_i \stackrel{\text{def}}{=} (1-s_i)x_i + s_i y_i \in K_2.$$

Further, for $i = 1, 2$, define segments with endpoints \bar{x}_i, \bar{y}_i ,

$$J_i \stackrel{\text{def}}{=} \{(1-t)\bar{x}_i + t\bar{y}_i : t \in [0, 1]\}.$$

Then we get that $J_i \cap K_1 = \{\bar{x}_i\}$, $J_i \cap K_2 = \{\bar{y}_i\}$ and $J_i \cap J_2 = \emptyset$ (since $J_i \subset I_i$ which were disjoint). Further

$$(3.1) \quad d(\bar{x}_i, \bar{y}_i) \leq d(x_i, y_i) < \alpha.$$

By [11, Theorem 62.V.6] there exists an open connected region G whose boundary is contained in $K_1 \cup K_2 \cup J_1 \cup J_2$. Since $K_1 \cup K_2 \subset f^{-1}(0)$, and by (3.1), we conclude that for all $x \in \text{Bd}(G)$,

$$(3.2) \quad |f(x)| \leq \Omega\left(\frac{\alpha}{2}\right).$$

Let $x_0 \in G$ be such that for $i = 1, 2$

$$d(x_0, \bar{x}_i) \geq \frac{1}{2}d(\bar{x}_1, \bar{x}_2).$$

Such a point x_0 exists in G since G is open and connected and thus G is path-connected. By Lemma 3.4 there exists a curve $\phi : [0, \infty) \rightarrow \mathbb{R}^2$ with Lipschitz constant one, $\phi(0) = x_0$

and $f(\phi(t)) = f(x_0) + t \operatorname{sign}(f(x_0))$. Since this curve is clearly unbounded, there exists $\tau > 0$ so that $\phi(\tau) \in \operatorname{Bd}(G)$. Then, by (3.2) and since ϕ is Lipschitz with constant one,

$$\begin{aligned} \Omega\left(\frac{\alpha}{2}\right) &\geq |f(\phi(\tau))| \geq \tau \geq \|\phi(\tau) - \phi(0)\| = \|\phi(\tau) - x_0\| \\ &\geq d(x_0, J_1 \cup J_2) \geq \min_{i=1,2} (d(x_0, \bar{x}_i)) - \alpha \\ &\geq \frac{1}{2}d(\bar{x}_1, \bar{x}_2) - \alpha. \end{aligned}$$

Thus

$$d(\bar{x}_1, \bar{x}_2) \leq 2\Omega\left(\frac{\alpha}{2}\right) + 2\alpha,$$

and

$$d(x_1, x_2) \leq d(x_1, \bar{x}_1) + d(\bar{x}_1, \bar{x}_2) + d(\bar{x}_2, x_2) \leq 2\Omega\left(\frac{\alpha}{2}\right) + 4\alpha. \quad \square$$

Proposition 3.5. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$, $f^{-1}(t)$ is hereditarily locally connected.*

Proof. Without loss of generality we will assume that f is a co-Lipschitz uniformly continuous map with a co-Lipschitz constant 1, and that $t = 0$. By Theorem 3.2, it is enough to show that $f^{-1}(0)$ does not contain a continuum of convergence.

Suppose for contradiction that K_0 is a continuum of convergence in $f^{-1}(0)$ and let $x_1, x_2 \in K_0$, and $\beta \stackrel{\text{def}}{=} d(x_1, x_2) > 0$. Let $[K_i]_{i=1}^{\infty}$ be the sequence of mutually disjoint subcontinua of $f^{-1}(0)$ with $\bigcup_i K_i \cap K_0 = \emptyset$ and $\lim [K_i]_i = K_0$. Then, by the definition of the limit (see also [23, Theorem I.(7.2)]), for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ so that for all $x \in K_0$

$$d(x, K_n) < \varepsilon.$$

Thus for $i = 1, 2$ there exists $y_i \in K_n$ with

$$d(x_i, y_i) < \varepsilon.$$

Hence, by Lemma 3.3, $d(x_1, x_2) \leq 2\Omega(\varepsilon/2) + 4\varepsilon$. Since ε is arbitrary and $\lim_{r \rightarrow 0} \Omega(r) = 0$, we conclude that $d(x_1, x_2) = 0$ which contradicts the fact that K_0 is a nontrivial subcontinuum. \square

4. FIRST DESCRIPTION OF THE STRUCTURE OF LEVEL SETS

The aim of this section is to obtain a characterization of the form of any closed, hereditarily locally connected, locally compact, connected set with no end points and containing no simple closed curve, and to apply it to describe the structure of level sets of co-Lipschitz uniformly continuous mappings $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ (Theorem 4.11 and Remark 4.13). For that we will need the notions of a dendrite, an order of a point, an end point and a cut point of a topological space M . We recall their definitions below.

Definition 4.1. [23, Chapter III], [11, Chapter VI, §51] Let M be a space and \mathfrak{n} a cardinal number. We say that a point $x \in M$ is of *order* $\leq \mathfrak{n}$ in M provided that for any neighborhood V of x in M , there exists a neighborhood U of x in M with $U \subset V$ and $\operatorname{card}(\operatorname{Bd}(U)) \leq \mathfrak{n}$.

A point $x \in M$ which is of order one in M will be called an *end point* of M .

Definition 4.2. [23, Section III.1], [11, Definition 47.VIII.2] If M is a connected set and p is a point of M such that the set $M \setminus \{p\}$ is not connected, then p is called a *cut point* of M .

Definition 4.3. [23, Section V.1] A continuum M is called a *dendrite* (or an *acyclic curve*) provided that M is locally connected and contains no simple closed curve.

Dendrites constitute a very important class of continua, and they have been extensively studied. We recall here a couple of important properties of dendrites, that we will use.

Theorem 4.4. [23, V.(1.1) and V.(1.2)] *Let M be a continuum. The following statements are equivalent:*

- (1) M is a dendrite.
- (2) Every point of M is either a cut point or an end point.
- (3) M is locally connected and one and only one arc exists between any two points in M .

We observe here a simple property of end points which we state in the form of a lemma for an easy reference:

Lemma 4.5. *Let M be a closed connected, locally connected subset of \mathbb{R}^n . Suppose that p is an end point of the subset $B \subset M$ such that:*

- (a) *there exists an open set $U \subset \mathbb{R}^n$ so that $p \in U$ and $M \setminus B \subset \mathbb{R}^n \setminus \bar{U}$; or*
- (b) *B is a component of $M \setminus A$ for some subcontinuum A of M .*

Then p is an end point of M .

Proof. For (a) let V be any neighborhood of p . Then, since p is an end point of B , there exists an open set $V_1 \subset V \cap U$ so that $p \in V_1$ and $\text{card}(\text{Bd}(V_1) \cap B) = 1$. Since $M \setminus B \subset \mathbb{R}^n \setminus \bar{U}$ we get that $(M \setminus B) \cap \bar{V}_1 = \emptyset$ and thus $\text{Bd}(V_1) \cap M = \text{Bd}(V_1) \cap B$, which ends the proof of part (a).

For (b) let V be any neighborhood of p . Since M is locally connected, there exists an open set $V_1 \subset V \setminus A$ so that $V_1 \cap M$ is connected. Since $p \in B$ and B is a connected component of $M \setminus A$ we conclude that $V_1 \cap M = V_1 \cap B$. Thus, since p is an end point of B , there exists an open set $V_2 \subset V_1$ so that $p \in V_2$, $\bar{V}_2 \subset V_1$, $\text{card}(\text{Bd}(V_2) \cap B) = 1$ and $\text{Bd}(V_2) \cap B = \text{Bd}(V_2) \cap M$, which ends the proof of part (b). \square

Our first observation concerning the structure of level sets of co-Lipschitz uniformly continuous mappings is the following:

Corollary 4.6. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$ and every subcontinuum M of $f^{-1}(t)$, M is a dendrite.*

Proof. It is easy to see that when f is a co-Lipschitz uniformly continuous map then for all t , $f^{-1}(t)$ cannot contain a simple closed curve. Indeed, since f is co-Lipschitz, $f^{-1}(t)$ has empty interior and if a simple closed curve C was contained in $f^{-1}(t)$ then $\mathbb{R}^2 \setminus f^{-1}(t)$ would have a bounded component A contained in the region inside the curve C . But then $f(A) = (t, \infty)$ or $(-\infty, t)$, which is impossible since \bar{A} is compact and f is continuous.

Further, by Proposition 3.5, every subcontinuum M of $f^{-1}(t)$ is locally connected. Thus M is a dendrite. \square

Our next goal is to show that every $f^{-1}(t)$ is of a particularly simple form, that every point of $f^{-1}(t)$ is of finite order and only finitely many points in $f^{-1}(t)$ have order bigger than 2. Thus we will show that $f^{-1}(t)$ has a graph structure. Recall the following

Definition 4.7. (cf. e.g. [23, Section X.1]) A set A is called a (finite linear) *graph* provided A is the union of a finite set V of points, called *vertices*, and a finite number of open mutually disjoint arcs $\alpha_1, \alpha_2, \dots, \alpha_n$, called *edges*, so that the two end points of each edge α_i are

distinct and belong to V . A graph which contains no simple closed curve is called a *tree* or an *acyclic graph* (see e.g. [14, Definition 9.25]).

We start from the following:

Proposition 4.8. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and let K be a component of $f^{-1}(t)$ for some $t \in \mathbb{R}$. Then every point of K is a cut point, i.e. $f^{-1}(t)$ has no end points.*

Proof. Suppose that $x \in K$ is not a cut point of K . Then by Corollary 4.6 and Theorem 4.4, x is an endpoint of K . It follows from [22, Theorem 26] (cf. also [5, Proof of Theorem 27], where this fact is attributed to R.G. Lubben), that if $x \in K$ is an endpoint of K then x belongs to the boundary of exactly one component of $\mathbb{R}^2 \setminus K$. But then, since $\mathbb{R}^2 \setminus f^{-1}(t)$ and thus also $\mathbb{R}^2 \setminus K$ have finite number of components, there exists a neighborhood U of x so that U intersects exactly one component of $\mathbb{R}^2 \setminus K$. Hence $f(U) \subset (t, \infty)$ or $f(U) \subset (-\infty, t)$, which contradicts the fact that f is co-Lipschitz. \square

To finish the analysis of the structure of level sets $f^{-1}(t)$ we will need one more notion – the notion of a number of ends of an unbounded locally connected set.

Definition 4.9. [7, Definition 1.18] We say that a connected locally connected Hausdorff space W has at least k ends if there exists an open subspace $V \subseteq W$ with compact closure \overline{V} so that $W \setminus \overline{V}$ has at least k unbounded components. The space W has exactly k ends if W has at least k ends but not at least $k + 1$ ends. If W has exactly k ends we will write $\#e(W) = k$.

One should be careful not to confuse ends with end points. We think of ends, intuitively, as infinite ends of unbounded sets. In fact, there exist ways of making this intuition precise, by defining ends using homotopy classes of unbounded paths contained in the space W (see [7]), but we will not need this for our present purpose.

Clearly continua never have any ends, but unbounded locally connected sets may have some end points in addition to the fact that they always have at least one end.

If a locally connected space W has a finite number of connected components, $W = \bigcup_{j=1}^m C_j$, then we will use notation $\#e(W)$ to mean the sum of $\#e(C_j)$, i.e.

$$\#e(W) \stackrel{\text{def}}{=} \sum_{j=1}^m \#e(C_j).$$

We note here that it follows from the local connectivity of W , by [6, Theorem 3-9], that if V is an open subset of W with compact closure, then $W \setminus \overline{V}$ has at most a finite number of unbounded components. If W has exactly k ends then there exists an open subspace $V \subseteq W$ with compact closure so that $W \setminus \overline{V}$ has exactly k unbounded components. Moreover we have the following:

Proposition 4.10. [7, Proposition 1.20 and its proof] *For an unbounded connected locally connected closed space $W \subset \mathbb{R}^n$ with exactly k ends there exists an open set $U \subset \mathbb{R}^n$ with compact closure so that W can be expressed as*

$$W = W_0 \cup \bigcup_{j=1}^k \overline{W(j)},$$

where $W_0 = W \cap \overline{U}$ is connected and compact, sets $W(j)$, for $j = 1, \dots, k$, are connected components of $W \setminus U$ and each $W(j)$ has exactly one end.

As a corollary we obtain the main theorem of this section which describes the structure of level sets $f^{-1}(t)$.

Theorem 4.11. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$, every component K of $f^{-1}(t)$ has a representation of the form:*

$$K = K_0 \cup \bigcup_{j=1}^n K_j,$$

where $n \in \mathbb{N}$, $n = \#e(K)$, K_0 is a compact connected tree with exactly n endpoints, each K_j is a ray, that is a closed unbounded set homeomorphic with $[0, \infty)$, sets $\{K_j\}_{j=1}^n$ are mutually disjoint, for all j , $\text{card}(K_j \cap K_0) = 1$ and the unique point in the intersection $K_j \cap K_0$ is an end point of K_0 and of K_j .

Definition 4.12. We will use the term *unbounded finite graph* for the sets of the form described in Theorem 4.11.

Proof. Let K be a component of $f^{-1}(t)$ for some $t \in \mathbb{R}$. By Proposition 3.5 K is locally connected and thus, by [6, Theorem 3-9], K has exactly n ends for some $n \in \mathbb{N}$, $n \geq 1$. Thus, by Proposition 4.10, there exists an open set $U \subset \mathbb{R}^2$ with compact closure so that K can be expressed as

$$K = W_0 \cup \bigcup_{j=1}^n \overline{W(j)},$$

where $W_0 = K \cap \overline{U}$ is connected and compact, and sets $W(j) \subset K \setminus U$ are connected components of $K \setminus U$ and each $W(j)$ has exactly one end. We define for $j \in \{1, \dots, n\}$,

$$K_j \stackrel{\text{def}}{=} \overline{W(j)}.$$

First we notice that, since every subcontinuum of K is a dendrite, we have for all $j = 1, \dots, n$:

$$(4.1) \quad \text{card}(\overline{W(j)} \cap W_0) = 1.$$

To see this, assume, for contradiction, that for some $j \in \{1, \dots, n\}$, there exist two points $x_1 \neq x_2$ in $\overline{W(j)} \cap W_0$. Let V_1, V_2 be open disjoint neighborhoods of x_1, x_2 , respectively, so that sets $V_i \cap \overline{W(j)}$, for $i = 1, 2$, are arcwise connected; this is possible since $\overline{W(j)}$ are locally connected. For $i = 1, 2$, let $y_i \in V_i \cap W(j)$ and let $\eta_i : [0, 1] \rightarrow V_i \cap \overline{W(j)}$ be a path so that $\eta_i(0) = y_i$, $\eta_i(1) = x_i$. Set, for $i = 1, 2$,

$$t_i \stackrel{\text{def}}{=} \inf\{t \in [0, 1] : \eta_i(t) \in W_0\},$$

$$z_i \stackrel{\text{def}}{=} \eta_i(t_i) \in \overline{W(j)} \cap W_0,$$

$$\widehat{\eta}_i \stackrel{\text{def}}{=} \eta_i([0, t_i]).$$

Next, note that since $W(j)$ is open and locally connected, by [11, Theorem 50.I.2], there exists an arc $\alpha \subset W(j)$ with endpoints y_1 and y_2 . Now, let $\beta = \widehat{\eta}_1 \cup \alpha \cup \widehat{\eta}_2$. Then β is an arc with endpoints z_1 and z_2 , and $\beta \setminus \{z_1, z_2\} \subset W(j)$.

On the other hand, since $z_1, z_2 \in W_0$ and since, by Corollary 4.6, W_0 is a dendrite we conclude, by Theorem 4.4(3), that there exists an arc $\gamma \subset W_0$ with endpoints z_1 and z_2 .

Thus we obtained two arcs β and γ in K with $\beta \cap \gamma = \{z_1, z_2\}$ which contradicts Theorem 4.4(3) and ends the proof of (4.1).

Define, for $j = 1, \dots, n$, w_j to be the unique point, by (4.1), of the intersection $\overline{W(j)} \cap W_0$. We claim that

$$(4.2) \quad \overline{W(j)} = W(j) \cup \{w_j\}.$$

Indeed, since $W(j)$ is a component of $K \setminus U$, $\text{Bd}(W(j)) \subset \text{Bd}(U)$ that is:

$$\overline{W(j)} \subset W(j) \cup \overline{U}.$$

Since K is closed and $W_0 = K \cap \overline{U}$ we get

$$\overline{W(j)} \subset K \cap (W(j) \cup \overline{U}) = W(j) \cup W_0.$$

Thus (4.2) follows from (4.1) and the definition of w_j .

Now fix a point $w_0 \in W_0 \cap U$. By Corollary 4.6 and Theorem 4.4(3), for each $j \in \{1, \dots, n\}$, there exists exactly one arc $\sigma_j \subset W_0 \subset K$ with endpoints w_0 and w_j . Set

$$K_0 \stackrel{\text{def}}{=} \bigcup_{j=1}^n \sigma_j.$$

It is clear from the definition that K_0 is a compact connected graph with exactly n endpoints $\{w_1, \dots, w_n\}$, and, by Corollary 4.6, K_0 does not contain any simple closed curve, thus K_0 is a tree. Also, by (4.2), for all $j \in \{1, \dots, n\}$, $\overline{W(j)} \cap K_0 = \{w_j\}$, an end point of K_0 .

We will now show that $W_0 = K_0$.

Clearly $K_0 \subset W_0$, so suppose, for contradiction, that $W_0 \setminus K_0$ is nonempty and let A be the closure of a connected component of $W_0 \setminus K_0$. By [14, Theorem 6.8] there exists a non-cut point p of $K_0 \cup A$ so that $p \in A$ and p is a non-cut point of W_0 , thus p is an end point of W_0 . Since $p \in A$, we see that $p \notin K_0$ and $p \notin \{w_1, w_2, \dots, w_n\} = K_0 \cap \bigcup_{j=1}^n \overline{W(j)}$. By (4.2)

$$W_0 \cap \bigcup_{j=1}^n \overline{W(j)} \subseteq \{w_1, w_2, \dots, w_n\}.$$

Thus

$$p \notin \bigcup_{j=1}^n \overline{W(j)}, \quad \text{and} \quad p \in U.$$

Hence, by Lemma 4.5(a), since p is an end point of W_0 , p is also an end point of K , which is a contradiction with Proposition 4.8. Thus we have shown that $W_0 = K_0$.

To finish the proof of the theorem it is only left to show that for each $j \in \{1, \dots, n\}$, the set $W(j)$ is homeomorphic with the real line. To see this we will use the classical theorem of Ward [21], which characterizes the real line as a non-empty connected, locally connected, separable metric space which is cut by each of its points into exactly 2 components.

Clearly, each $W(j)$ is a non-empty, connected, locally connected, separable metric space; and, by Proposition 4.8, every point of $W(j)$ is a cut point of K , and thus also of $W(j)$. Suppose that there exists a point $x \in W(j)$ so that $W(j) \setminus \{x\}$ has 3 or more components. Since $W(j)$ has exactly one end, exactly one component of $W(j) \setminus \{x\}$ is unbounded. Thus

$W(j) \setminus \{x\}$, and therefore also $\overline{W(j)} \setminus \{x\} = (W(j) \cup \{w_j\}) \setminus \{x\}$, have at least 2 bounded components, say C_1, C_2 .

Let V be an open set with compact closure so that $\overline{C_1} \cup \overline{C_2} \subset V$. Then C_1, C_2 are components of $\overline{W(j)} \cap \overline{V}$. Without loss of generality we assume that $w_j \notin C_1$. Then, by [14, Corollary 5.9],

$$\overline{C_1} = C_1 \cup \{x\},$$

and, by [14, Theorem 6.8], there exists a non-cut point p of $\overline{C_1}$ so that $p \neq x$ and p is a non-cut point of $\overline{W(j)} \cap \overline{V}$. Hence, by Corollary 4.6 and Lemma 4.5(a), p is an end point of $\overline{W(j)} = W(j) \cup \{w_j\}$. Since $p \neq w_j$, by Lemma 4.5(b), p is an end point of K and we get a contradiction with Proposition 4.8. Thus we have shown that every point of $W(j)$ cuts $W(j)$ into exactly 2 components and hence $W(j)$ is homeomorphic to the real line and $\overline{W(j)} = W(j) \cup \{w_j\}$ is a ray, which ends the proof of the theorem. \square

Remark 4.13. As the reader has surely noticed, the above proof and hence also the conclusion of Theorem 4.11 is valid for any set K such that:

$K \subset \mathbb{R}^m$, $m \geq 2$, is a closed, hereditarily locally connected, locally compact, connected set with no end points and containing no simple closed curve.

Remark 4.14. There is an alternative way to prove Theorem 4.11. Instead of the fairly direct and self-contained proof presented above, one could use results of Shimrat [17, 18] and Stone [20] who (among others) studied the structure of sets whose every point is a cut point. In particular Shimrat [18] fully described locally connected sets whose every point is a cut point and this characterization when refined with [6, Theorem 3-9] and the assumption that the set is closed also gives the statement of Theorem 4.11.

On the other hand, Stone [20] gave a characterization of finite linear graphs, from which it follows easily that a dendrite is a (finite linear) tree if and only if it has a finite number of end points. This result of Stone has been, using different methods, reproved and strengthened by Nadler [15] (cf. also [14, Theorem 9.24]). It is clear that this characterization is closely related with Theorem 4.11 and indeed it is possible to obtain a proof of Theorem 4.11 using these results.

However we felt that following either of these two routes of reasoning would be technically more complicated than the presented direct proof.

5. NUMBER AND FORM OF COMPONENTS OF LEVEL SETS

In this section we present an exact characterization of the form of level sets of a co-Lipschitz uniformly continuous map f from \mathbb{R}^2 to \mathbb{R} (Theorem 5.1), which significantly refines Theorem 4.11. In particular, we obtain an affirmative answer to the question posed in [8] whether the number of components of level sets $f^{-1}(t)$ or of $\mathbb{R}^2 \setminus f^{-1}(t)$ are constant after excluding finitely many values of t . We begin with the statement of our main characterization theorem.

We will use the notation $\#c(W)$ to denote the number of components of the set W .

Theorem 5.1. *For every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r \rightarrow 0} \Omega(r) = 0$, there exists a natural number $M = M(c, \Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω , there exists a natural number $n = n(f) \leq M$ and a finite subset T_f of \mathbb{R} , with $\text{card}(T_f) \leq n - 1$, so that:*

(1) for all $t \in \mathbb{R}$,

$$\#e(f^{-1}(t)) = 2n,$$

that is there exists $R_0 \in \mathbb{R}$ so that for every $R > R_0$, $f^{-1}(t) \setminus B(0, R)$ has exactly $2n$ unbounded components.

Moreover, if $\{C_{R,i}\}_{i=1}^{2n}$ are the unbounded components of $f^{-1}(t) \setminus B(0, R)$, then for all $i \neq j$,

$$\lim_{R \rightarrow \infty} d(C_{R,i}, C_{R,j}) = \infty;$$

(2) for all $t \in \mathbb{R} \setminus T_f$,

(a) $\#c(f^{-1}(t)) = n$,

(b) $\#c(\mathbb{R}^2 \setminus f^{-1}(t)) = n + 1$,

(c) each component of $f^{-1}(t)$ is homeomorphic with the real line and separates the plane into exactly 2 components;

(3) for all $t_i \in T_f$,

(a) $\#c(f^{-1}(t_i)) < n$,

(b) $\#c(\mathbb{R}^2 \setminus f^{-1}(t_i)) = 2n + 1 - \#c(f^{-1}(t_i)) \in (n + 1, 2n)$,

(c) each component of $f^{-1}(t_i)$ is an unbounded finite graph, i.e. has the form described in Theorem 4.11

Remark 5.2. Theorem 5.1 is analogous to a result of Johnson, Lindenstrauss, Preiss and Schechtman [8], who proved that for every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r \rightarrow 0} \Omega(r) = 0$, there exists a natural number $M = M(c, \Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω , there exists a natural number $n = n(f) \leq M$ and a polynomial P_f with degree equal to n , so that $f = P_f \circ h_f$, where h_f is a homeomorphism of \mathbb{R}^2 . Hence there exists a finite set $T_f \subset \mathbb{R}^2$ with $\text{card}(T_f) \leq n \leq M$, so that for all $t \in \mathbb{R}^2 \setminus T_f$, $\text{card}(f^{-1}(t)) = n$ and for all $t_i \in T_f$, $\text{card}(f^{-1}(t_i)) < n$, analogously with parts (2a) and (3a) of Theorem 5.1.

For Lipschitz quotient maps from \mathbb{R}^2 to \mathbb{R}^2 , Maleva [12] studied the dependence of the number $M(c, L)$ on the Lipschitz and co-Lipschitz constants L and c . Maleva proved in particular that there exists a scale $0 < \dots < \varrho_2^{(m)} < \dots < \varrho_2^{(1)} < 1$ such that for any Lipschitz quotient mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the condition $c/L > \varrho_2^{(m)}$ implies that $\text{card}(f^{-1}(t)) \leq m$ for all $t \in \mathbb{R}^2$ (in fact this holds with $\varrho_2^{(m)} = 1/(m + 1)$) [12, Theorem 2].

It is natural to ask whether a similar scale exists for the numbers $M(c, \Omega)$ defined in Theorem 5.1. [After reading a preliminary version of this paper, Maleva proved the existence of such a scale. Namely she proved that if $c/L > \sin(\pi/(2n))$ then for all $t \in \mathbb{R}$, $\#c(f^{-1}(t)) < n$, [13]. A similar scale also exists for co-Lipschitz uniformly continuous maps [13].]

Remark 5.3. The estimate of the cardinality of the exceptional set T_f is best possible, in the sense that for any $n \in \mathbb{N}$ it is easy to construct examples of Lipschitz quotient mappings $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that $\text{card}(T_f) = n - 1$ and $\#c(f^{-1}(t)) = n$ for all $t \in \mathbb{R} \setminus T_f$. In Figure 5.1 below, we present sketches of examples of such functions for $n = 2, 3, 4$. In each sketch, level sets for different values of t are represented by different styles of lines (within limits set by the drawing program (XY-pic)), and the mapping f is the distance in the ℓ_1 -metric from the solid lines, which represent the preimage of 0, multiplied, in each component of the complement of the solid lines, by the sign indicated.

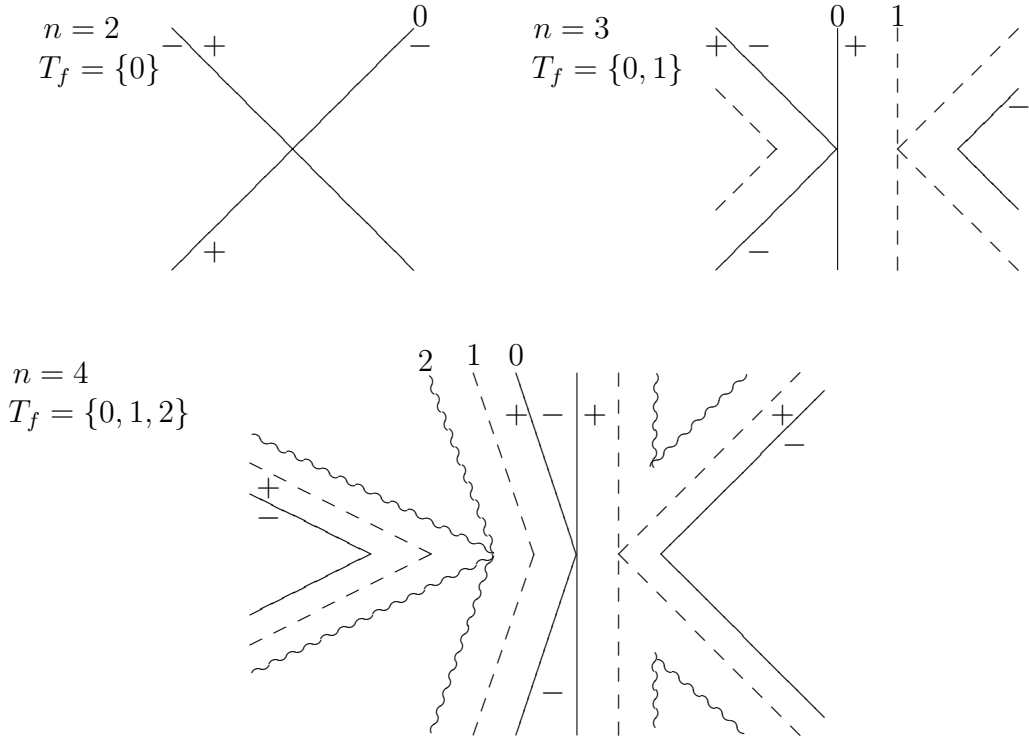


FIGURE 5.1.

For the proof of Theorem 5.1 we will need a large number of auxiliary results concerning the number of components of level sets $f^{-1}(t)$ and the end structure of boundaries of components of the complements of $f^{-1}(t)$. We start from a presentation of these results and postpone the proof of Theorem 5.1 to the end of this section.

Our first observation relates the number of components of $f^{-1}(t)$ with the number of components of $\mathbb{R}^2 \setminus f^{-1}(t)$.

Proposition 5.4. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and K be a connected component of the level set $f^{-1}(t)$. Then*

$$\#c(\mathbb{R}^2 \setminus K) = \#e(K).$$

In particular, $\#e(K) \geq 2$.

Moreover, each component of $\mathbb{R}^2 \setminus K$ is homeomorphic with \mathbb{R}^2 and its boundary is connected and has exactly 2 ends.

Proof. By Theorem 4.11, K is an unbounded finite graph, i.e. K has a representation of the form:

$$K = K_0 \cup \bigcup_{j=1}^n K_j,$$

where $n = \#e(K)$, K_0 is a compact connected tree, each K_j is a closed unbounded set homeomorphic with $[0, \infty)$, sets $\{K_j\}_{j=1}^n$ are mutually disjoint and $\text{card}(K_j \cap K_0) = 1$ for all j .

Let $\alpha(K_0)$ denote the number of vertices of K_0 and $\beta(K_0)$ the number of edges of K_0 . It is a basic fact of the graph theory (see e.g. [9, Theorem IV.9 (attributed to Listing 1862)]) that since K_0 is a tree we have

$$\alpha(K_0) = \beta(K_0) + 1.$$

Now let \overline{K} be the closure of K in the sphere \mathcal{S}_2 which is a one-point compactification of \mathbb{R}^2 . Then \overline{K} is a compact graph in \mathcal{S}_2 , and, by the well-known Euler's formula, we have

$$\alpha(\overline{K}) - \beta(\overline{K}) + \gamma(\overline{K}) = 2,$$

where $\alpha(\overline{K})$ is the number of vertices of \overline{K} , $\beta(\overline{K})$ is the number of edges of \overline{K} , and $\gamma(\overline{K})$ is the number of components of $\mathcal{S}_2 \setminus \overline{K}$. We have:

$$\begin{aligned}\alpha(\overline{K}) &= \alpha(K_0) + 1, \\ \beta(\overline{K}) &= \beta(K_0) + n.\end{aligned}$$

Thus

$$\begin{aligned}\#c(\mathbb{R}^2 \setminus K) &= \gamma(\overline{K}) = 2 - \alpha(\overline{K}) + \beta(\overline{K}) \\ &= 2 - \alpha(K_0) - 1 + \beta(K_0) + n \\ &= n = \#e(K),\end{aligned}$$

as claimed (since the number of components of $\mathbb{R}^2 \setminus K$ is equal to the number of components of $\mathcal{S}_2 \setminus \overline{K}$).

Since, by Corollary 2.5, K separates the plane, thus $\#e(K) = \#c(\mathbb{R}^2 \setminus K) \geq 2$.

Further, by [11, Theorem 61.II.4], the boundary of every component of $\mathcal{S}_2 \setminus \overline{K}$ is a simple closed curve. Thus by [11, Theorem 61.V.1] each component of $\mathcal{S}_2 \setminus \overline{K}$, and therefore also of $\mathbb{R}^2 \setminus K$, is homeomorphic with \mathbb{R}^2 . Since ∞ belongs to the boundary of every component of $\mathcal{S}_2 \setminus \overline{K}$, and since this boundary is a simple closed curve, we conclude that the order of ∞ , as a point of the boundary of any component of $\mathcal{S}_2 \setminus \overline{K}$, is equal to two and thus this boundary has exactly 2 ends, and it is connected as a subset of \mathbb{R}^2 . \square

Proposition 5.4 has two useful consequences.

Corollary 5.5. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and A be a component of $\mathbb{R}^2 \setminus f^{-1}(t)$. Then*

$$\#e(\text{Bd}(A)) = 2\#c(f^{-1}(t) \cap \text{Bd}(A)).$$

Proof. Let $\{K_i\}_{i=1}^n$ be components of $f^{-1}(t)$. If $\#c(f^{-1}(t) \cap \text{Bd}(A)) = 1$, then A is a component of, say, $\mathbb{R}^2 \setminus K_1$ and, by Proposition 5.4,

$$\#e(\text{Bd}(A)) = 2.$$

We suppose, for the induction, that if $\#c(f^{-1}(t) \cap \text{Bd}(A)) \leq k$, i.e. if A is a component $\mathbb{R}^2 \setminus \bigcup_{j=1}^k K_j$, then

$$(5.1) \quad \#e(\text{Bd}(A)) = 2\#c(f^{-1}(t) \cap \text{Bd}(A)).$$

Let B be a component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^{k+1} K_j$, say $B = A \cap C$, where A is a component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^k K_j$ and C is a component of $\mathbb{R}^2 \setminus K_{k+1}$.

If $K_{k+1} \not\subset A$, then, by the connectedness of K_{k+1} , $K_{k+1} \cap A = \emptyset$ and either $A \subset C$ or $A \cap C = \emptyset$. Since $B \neq \emptyset$, we obtain that $B = A$ and B is a component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^k K_j$ and, by the inductive hypothesis, there is nothing to prove.

Thus, without loss of generality, we assume that $K_{k+1} \subset A$ and $\text{Bd}(C) \subset K_{k+1} \subset A$. Then

$$(5.2) \quad \text{Bd}(C) \subset \text{Bd}(A \cap C).$$

Similarly, if $\bigcup_{j=1}^k K_j \not\subset C$ then by the connectedness of sets $\{K_j\}_{j=1}^k$, there exists $i_0 \leq k$ so that $K_{i_0} \not\subset C$ and thus $K_{i_0} \cap C = \emptyset$. Hence any component of $\mathbb{R}^2 \setminus K_{i_0}$ is either disjoint with C , or contains C . Thus B can be represented as an intersection of components of $\{\mathbb{R}^2 \setminus K_j\}_{j=1, j \neq i_0}^{k+1}$ and, by the inductive hypothesis, we are done.

Thus, without loss of generality, we assume that $\bigcup_{j=1}^k K_j \subset C$. Hence, as before, $\text{Bd}(A) \subset \bigcup_{j=1}^k K_j \subset C$ and

$$(5.3) \quad \text{Bd}(A) \subset \text{Bd}(A \cap C).$$

By [10, Formula 6.II(8)], we have

$$(5.4) \quad \text{Bd}(A \cap C) \subset \text{Bd}(A) \cup \text{Bd}(C).$$

Combining (5.2), (5.3) and (5.4) we get

$$\text{Bd}(A \cap C) = \text{Bd}(A) \cup \text{Bd}(C),$$

and, since $\text{Bd}(A) \cap \text{Bd}(C) = \emptyset$, we conclude that

$$\#e(\text{Bd}(A \cap C)) = \#e(\text{Bd}(A)) + \#e(\text{Bd}(C)).$$

Thus, by (5.1) and (5.2),

$$\begin{aligned} \#e(\text{Bd}(B)) &= 2\#c(f^{-1}(t) \cap \text{Bd}(A)) + 2 \\ &= 2\#c(f^{-1}(t) \cap (\text{Bd}(A) \cup K_{k+1})) \\ &= 2\#c(f^{-1}(t) \cap \text{Bd}(B)), \end{aligned}$$

which ends the proof. \square

Corollary 5.6. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and $t \in \mathbb{R}$. Then*

$$\#c(f^{-1}(t)) + \#c(\mathbb{R}^2 \setminus f^{-1}(t)) = \#e(f^{-1}(t)) + 1.$$

Proof. By Theorems 1.1 and 2.4 both $\mathbb{R}^2 \setminus f^{-1}(t)$ and $f^{-1}(t)$ have finite number of components. Denote $l = \#c(f^{-1}(t))$ and let $\{K_j\}_{j=1}^l$ be the components of $f^{-1}(t)$.

By Proposition 5.4, $\#c(\mathbb{R}^2 \setminus K_1) = \#e(K_1)$, and K_2 is contained in exactly one of the components of $\mathbb{R}^2 \setminus K_1$, say C . Since C is homeomorphic with \mathbb{R}^2 , again by Proposition 5.4 we conclude that

$$\#c(C \setminus K_2) = \#e(K_2),$$

and thus

$$\begin{aligned} \#c(\mathbb{R}^2 \setminus (K_1 \cup K_2)) &= \#c(\mathbb{R}^2 \setminus K_1) - 1 + \#c(\mathbb{R}^2 \setminus K_2) \\ &= \#e(K_1) + \#e(K_2) - 1. \end{aligned}$$

Proceeding by induction we get

$$\begin{aligned} \#c(\mathbb{R}^2 \setminus f^{-1}(t)) &= \#c\left(\mathbb{R}^2 \setminus \left(\bigcup_{j=1}^l K_j\right)\right) \\ &= \sum_{j=1}^l \#e(K_j) - (l - 1) \\ &= \#e(f^{-1}(t)) + 1 - \#c(f^{-1}(t)), \end{aligned}$$

which ends the proof of the corollary. \square

As a consequence of Theorems 1.1, 2.4 and Corollary 5.6 we immediately obtain:

Corollary 5.7. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$, the number of ends of $f^{-1}(t)$ is finite and bounded by a function depending only on the co-Lipschitz constant of f and its modulus of uniform continuity.*

Our next goal is to show that the number of ends of $f^{-1}(t)$ is independent of t . To achieve this we first prove that different ends of level sets $f^{-1}(t)$ are “infinitely far away” from each other, as on Figure 1.2 in the Introduction. To state this precisely we will use the notation $d(X, Y)$ to denote the distance between sets X, Y i.e.

$$d(X, Y) \stackrel{\text{def}}{=} \inf\{d(x, y) : x \in X, y \in Y\}.$$

Proposition 5.8. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and $t \in \mathbb{R}$. If $f^{-1}(t)$ has l components $\{K^i(t)\}_{i=1}^l$, and each component $K^i(t)$ has the following representation of the form described in Theorem 4.11,*

$$K^i(t) = K_0^i(t) \cup \bigcup_{j=1}^{n(i)} K_j^i(t),$$

then, for all $i_1, i_2 \in \{1, \dots, l\}$, $j_1 \in \{1, \dots, n(i_1)\}$, $j_2 \in \{1, \dots, n(i_2)\}$, if the ordered pairs $(i_1, j_1), (i_2, j_2)$ are not the same, then

$$\lim_{R \rightarrow \infty} d(K_{j_1}^{i_1}(t) \setminus B(0, R), K_{j_2}^{i_2}(t) \setminus B(0, R)) = \infty.$$

Proof. This result follows almost immediately from Lemma 3.3. Without loss of generality we assume that the co-Lipschitz constant of f is 1 and let $\Omega(\cdot)$ be the modulus of uniform continuity of f . For any $R \in \mathbb{R}_+$ denote

$$d_R = d(K_{j_1}^{i_1}(t) \setminus B(0, R), K_{j_2}^{i_2}(t) \setminus B(0, R)).$$

Clearly $d_{R_1} \geq d_{R_2}$ when $R_1 \geq R_2$, thus, if $\lim_{R \rightarrow \infty} d_R \neq \infty$ then there exists $\alpha \in \mathbb{R}_+$ so that for all $R \in \mathbb{R}$,

$$(5.5) \quad d_R \leq \alpha.$$

Fix $x_1 \in K_{j_1}^{i_1}(t)$ and $y_1 \in K_{j_2}^{i_2}(t)$ so that

$$d(x_1, y_1) \leq \alpha.$$

Set

$$\tilde{R} \stackrel{\text{def}}{=} \|x_1\| + 2\Omega\left(\frac{\alpha}{2}\right) + 4\alpha + 1.$$

Then, by (5.5), there exist $x_2 \in K_{j_1}^{i_1}(t) \setminus B(0, \tilde{R})$ and $y_2 \in K_{j_2}^{i_2}(t) \setminus B(0, \tilde{R})$ with

$$d(x_2, y_2) \leq \alpha.$$

Since, for $\nu = 1, 2$, the sets $K_{j_\nu}^{i_\nu}(t)$ are connected subsets of $f^{-1}(t)$, there exist arcs $\sigma_\nu \subset K_{j_\nu}^{i_\nu}(t) \subset f^{-1}(t)$ with endpoints x_ν, y_ν . Since $(i_1, j_1) \neq (i_2, j_2)$, the arcs $\sigma_\nu, \nu = 1, 2$, are disjoint subcontinua of $f^{-1}(t)$, and hence, by Lemma 3.3,

$$d(x_1, x_2) \leq 2\Omega\left(\frac{\alpha}{2}\right) + 4\alpha.$$

But

$$d(x_1, x_2) \geq \left| \|x_2\| - \|x_1\| \right| \geq \tilde{R} - \|x_1\| = 2\Omega\left(\frac{\alpha}{2}\right) + 4\alpha + 1,$$

and the resulting contradiction ends the proof of Proposition 5.8. \square

As an immediate corollary we obtain the following two facts which we state here for an easy reference.

Corollary 5.9. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and $f^{-1}(t)$ have l components $\{K^i(t)\}_{i=1}^l$. Then for all $i_1, i_2 \leq l$, $i_1 \neq i_2$, we have*

$$d(K^{i_1}(t), K^{i_2}(t)) > 0.$$

Proof. This follows immediately from Proposition 5.8. Continuing the same notation as above, let $R_0 \in \mathbb{R}$ be such that for all $j_1 \in \{1, \dots, n(i_1)\}$, $j_2 \in \{1, \dots, n(i_2)\}$,

$$d(K_{j_1}^{i_1}(t) \setminus B(0, R_0), K_{j_2}^{i_2}(t) \setminus B(0, R_0)) \geq 1.$$

Since $K^{i_\nu}(t) \cap \overline{B(0, R_0)}$, for $\nu = 1, 2$, are compact and disjoint we conclude that

$$d(K^{i_1}(t), K^{i_2}(t)) \geq \min(1, d) > 0,$$

as desired. □

Corollary 5.10. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$, $R \in \mathbb{R}_+$ and $\{C_{R,i}\}_i$ are a collection of unbounded components of $f^{-1}(t) \setminus B(0, R)$. Then, for all $i \neq j$,*

$$\lim_{R \rightarrow \infty} d(C_{R,i}, C_{R,j}) = \infty.$$

Remark 5.11. After reading a preliminary version of this paper, Maleva has strengthened the conclusion of Corollary 5.10. She proved [13], in the notation as above, that there exists a constant $\delta > 0$ depending only on the modulus of continuity of f and its co-Lipschitz constant, so that for every $t \in \mathbb{R}$ there exists $R(t) > 0$ so that for all $R > R(t)$ and all $i \neq j$,

$$d(C_{R,i}, C_{R,j}) \geq \delta R.$$

This has consequences not only for the topology, but also for the allowable geometric structure of $f^{-1}(t)$, e.g. $f^{-1}(t)$ cannot contain a parabola, see [13].

As a consequence of Proposition 5.8, we obtain three somewhat technical facts which will be important for our further arguments.

Lemma 5.12. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and $t_1, t_2 \in \mathbb{R}$. Then*

- (a) $\#e(f^{-1}(t_1)) = \#e(f^{-1}(t_2))$.
- (b) *If $t_1 > t_2$, A is a component of $f^{-1}(t_2, \infty)$ or $A = \mathbb{R}^2$ and $f^{-1}(t_1, \infty) \cap A = \bigcup_{\lambda=1}^n A_\lambda$, where A_λ are components of $f^{-1}(t_1, \infty) \cap A$, then*

$$\sum_{\lambda=1}^n \#e(\text{Bd}(A_\lambda)) = \#e(f^{-1}(t_1) \cap A).$$

- (c) *If $t_1 > t_2$ and A is a component of $f^{-1}(t_2, \infty)$, then*

$$\#e(f^{-1}(t_1) \cap A) = \#e(\text{Bd}(A)).$$

Proof. As before we assume without loss of generality that the co-Lipschitz constant of f is 1. By Corollary 5.6 (or a combination of earlier results in this paper) we know that both $f^{-1}(t_1)$

and $f^{-1}(t_2)$ have a finite number of ends. By Theorems 2.4 and 4.11, for $\nu = 1, 2$, $f^{-1}(t_\nu)$ can be represented as

$$(5.6) \quad f^{-1}(t_\nu) = \bigcup_{i=1}^{l_\nu} \left(K_0^i(t_\nu) \cup \bigcup_{j=1}^{n_\nu(i)} K_j^i(t_\nu) \right),$$

where $l_\nu, n_\nu(i) \in \mathbb{N}$ and $K_j^i(t_\nu)$ are mutually disjoint unbounded rays in \mathbb{R}^2 , and sets $K_0^i(t_\nu)$ are compact.

Further, for $\nu = 1, 2$, the number of distinct rays $K_j^i(t_\nu)$ is finite and equals the number of ends of $f^{-1}(t_\nu)$, i.e.

$$(5.7) \quad \text{card } \mathcal{R}(t_\nu) = \#e(f^{-1}(t_\nu)),$$

where $\mathcal{R}(t_\nu) \stackrel{\text{def}}{=} \{K_j^i(t_\nu) : i = 1, \dots, l_\nu, j = 1, \dots, n_\nu(i)\}$.

By Proposition 5.8, if Y_1, Y_2 are distinct rays of the same level set $f^{-1}(t_\nu)$, where $\nu \in \{1, 2\}$, then

$$\lim_{R \rightarrow \infty} d(Y_1 \setminus B(0, R), Y_2 \setminus B(0, R)) = \infty.$$

Thus, there exists $R_0 \in \mathbb{R}$ so that, for $\nu = 1, 2$, and for all $i \leq l_\nu$,

$$(5.8) \quad K_0^i(t_\nu) \subset B(0, R_0 - 1),$$

and so that, for any distinct rays Y_1, Y_2 of the same level set $f^{-1}(t_\nu)$, where $\nu = 1$ or 2 ,

$$(5.9) \quad d(Y_1 \setminus B(0, R_0), Y_2 \setminus B(0, R_0)) \geq 1 + 4|t_1 - t_2|.$$

On the other hand, since f is co-Lipschitz with constant 1, for every $x \in f^{-1}(t_1)$ and for every $r > 0$,

$$B(f(x), r) = (t_1 - r, t_1 + r) \subset f(B(x, r)).$$

Since $t_2 \in (t_1 - r, t_1 + r)$ when $r = 2|t_1 - t_2|$, we conclude that

$$(5.10) \quad \text{for every } x \in f^{-1}(t_1) \text{ there exists } y \in B(x, 2|t_1 - t_2|) \cap f^{-1}(t_2).$$

Now let $X \in \mathcal{R}(t_1)$ be a ray from the representation of $f^{-1}(t_1)$ described in (5.6), and let $x \in X$ be such that $\|x\| \geq R_0 + 2|t_1 - t_2|$. Then by (5.10) and (5.8) there exists at least one ray $Y \in \mathcal{R}(t_2)$ so that $d(x, Y \setminus B(0, R_0)) < 2|t_1 - t_2|$.

Suppose that there exist two distinct rays $Y_1, Y_2 \in \mathcal{R}(t_2)$ so that for $\alpha = 1, 2$,

$$d(x, Y_\alpha \setminus B(0, R_0)) < 2|t_1 - t_2|.$$

But then we would have

$$d(Y_1 \setminus B(0, R_0), Y_2 \setminus B(0, R_0)) < 4|t_1 - t_2|,$$

which contradicts (5.9).

Thus we have described a one-to-one map γ from the set of rays of $f^{-1}(t_1)$ into the set of rays of $f^{-1}(t_2)$, i.e. from $\mathcal{R}(t_1)$ into $\mathcal{R}(t_2)$, and γ operates in such a way that for every $X \in \mathcal{R}(t_1)$ and for every $x \in X$ with $\|x\| \geq R_0 + 2|t_1 - t_2|$ we have

$$(5.11) \quad d(x, \gamma(X) \setminus B(0, R_0)) < 2|t_1 - t_2|.$$

Since γ is one-to-one, by (5.7) and by symmetry, we have

$$\#e(f^{-1}(t_1)) = \#e(f^{-1}(t_2)),$$

which ends the proof of part (a).

Moreover, we conclude that γ is a bijection from $\mathcal{R}(t_1)$ onto $\mathcal{R}(t_2)$.

To prove part (b) we keep the same notation as above and we note that if $f^{-1}(t_1, \infty) \cap A = \bigcup_{\lambda=1}^n A_\lambda$, where A_λ are components of $f^{-1}(t_1, \infty) \cap A$, then

$$(5.12) \quad f^{-1}(t_1) \cap A = \bigcup_{\lambda=1}^n \text{Bd}(A_\lambda),$$

and therefore

$$\#e(f^{-1}(t_1) \cap A) = \#e\left(\bigcup_{\lambda=1}^n \text{Bd}(A_\lambda)\right).$$

Denote by $\mathcal{R}_A(t_1)$ the set of rays of $f^{-1}(t_1)$ contained in A , i.e.

$$\mathcal{R}_A(t_1) \stackrel{\text{def}}{=} \{K_j^i(t_1) \subset \mathcal{R}(t_1) : K_j^i(t_1) \subset A\}.$$

Since each ray $K_j^i(t_1)$ has exactly one end, we get

$$(5.13) \quad \#e(f^{-1}(t_1) \cap A) = \text{card}(\mathcal{R}_A(t_1)).$$

By Theorem 4.11, for $R_0 \in \mathbb{R}_+$ defined above, and for each $K_j^i(t_1) \in \mathcal{R}_A(t_1)$, the set $K_j^i(t_1) \setminus B(0, R_0)$ has a unique unbounded component; we will denote these components by $\{X_\alpha : \alpha = 1, \dots, \#e(f^{-1}(t_1) \cap A)\}$. Note that, by Theorem 4.11, each X_α is homeomorphic with $[0, \infty)$. We will show that

$$(5.14) \quad \text{for each } \alpha \leq \#e(f^{-1}(t_1) \cap A) \text{ there exists a unique } \lambda(\alpha) \leq n \text{ with } X_\alpha \subset \text{Bd}(A_{\lambda(\alpha)}).$$

Once (5.14) is established, part (b) follows easily. Indeed, by (5.14) we can define sets

$$E_\lambda \stackrel{\text{def}}{=} \{\alpha \leq \#e(f^{-1}(t_1) \cap A) : X_\alpha \subset \text{Bd}(A_\lambda)\},$$

and sets E_λ are disjoint. Note that $\text{card}(E_\lambda) = \#e(\text{Bd}(A_\lambda))$. Moreover, by (5.12),

$$\bigcup_{\lambda=1}^n \text{Bd}(A_\lambda) \supset \{X_\alpha : \alpha = 1, \dots, \#e(f^{-1}(t_1) \cap A)\},$$

so $\bigcup_{\lambda=1}^n E_\lambda = \{1, \dots, \#e(f^{-1}(t_1) \cap A)\}$ and thus

$$\#e(f^{-1}(t_1) \cap A) = \sum_{\lambda=1}^n \text{card}(E_\lambda) = \sum_{\lambda=1}^n \#e(\text{Bd}(A_\lambda)),$$

as desired.

To prove (5.14), note that by (5.12) and since sets $\text{Bd}(A_\alpha)$ are closed, for each $\alpha \leq \#e(f^{-1}(t_1) \cap A)$:

$$\overline{X_\alpha} \subset \bigcup_{\lambda=1}^n \text{Bd}(A_\lambda).$$

Thus

$$\overline{X_\alpha} = \bigcup_{\lambda=1}^n (\overline{X_\alpha} \cap \text{Bd}(A_\lambda)).$$

Since $\overline{X_\alpha}$ is connected and sets $\overline{X_\alpha} \cap \text{Bd}(A_\lambda)$ are closed, we conclude that either there exists a unique $\lambda(\alpha)$ so that, for all $\lambda \neq \lambda(\alpha)$,

$$\overline{X_\alpha} \cap \text{Bd}(A_\lambda) = \emptyset,$$

and in this case part (b) holds, or otherwise there exist $\lambda_1, \lambda_2 \leq n, \lambda_1 \neq \lambda_2$ so that

$$(5.15) \quad (\overline{X_\alpha} \cap \text{Bd}(A_{\lambda_1})) \cap (\overline{X_\alpha} \cap \text{Bd}(A_{\lambda_2})) \neq \emptyset.$$

But this alternative leads to a contradiction. Indeed, suppose that

$$x \in \overline{X_\alpha} \cap \text{Bd}(A_{\lambda_1}) \cap \text{Bd}(A_{\lambda_2}).$$

Since $\overline{X_\alpha}$ is a ray, i.e. a homeomorph of $[0, \infty)$, contained in one of the rays $\{K_j^i(t_1)\}_{i,j}$ of $f^{-1}(t_1) \cap A$, and since, by (5.8), $\overline{X_\alpha}$ is disjoint with all sets $\{K_0^i(t_1)\}_i$ we conclude that the order of the point x in $f^{-1}(t_1)$ is equal to 2. Hence, by Definition 4.1, for every neighborhood V of x , there exists a neighborhood of x with $U \subset V$ and so that $\text{card}(\text{Bd}(U) \cap f^{-1}(t_1)) = 2$. By (5.9) and since $f^{-1}(t_1)$ is locally connected, we can choose $U \subset A$ so that $x \in U$, $\text{Bd}(U)$ is a simple curve, $\text{Bd}(U) \cap f^{-1}(t_1) = \{x_1, x_2\}$ and x belongs to an arc contained in $f^{-1}(t_1)$ with endpoints x_1 and x_2 . Then, by the Theorem About The θ -Curve [11, Theorem 61.II.2], $U \setminus f^{-1}(t_1)$ has exactly two components, and consequently x belongs to the boundary of exactly two components of $\mathbb{R}^2 \setminus f^{-1}(t_1)$. Since f is co-Lipschitz, it is not possible that both of these components are contained in $f^{-1}(t_1, \infty)$, which contradicts (5.15) and ends the proof of (5.14) and of part (b).

For part (c), let $\{A_\nu\}_{\nu=1}^m$ denote the collection of components of $f^{-1}(t_2, \infty)$. As above, let $\mathcal{R}_{A_\nu}(t_1)$ denote the set of rays of $f^{-1}(t_1)$ contained in A_ν . By (5.13), for any $\nu \in \{1, \dots, m\}$,

$$(5.16) \quad \#e(f^{-1}(t_1) \cap A_\nu) = \text{card}(\mathcal{R}_{A_\nu}(t_1)).$$

Let $\gamma : \mathcal{R}(t_1) \rightarrow \mathcal{R}(t_2)$ be the map defined in part (a). We will show that for all $\nu \in \{1, \dots, m\}$,

$$(5.17) \quad X \in \mathcal{R}_{A_\nu}(t_1) \implies \gamma(X) \subset \text{Bd}(A_\nu).$$

Indeed, let $X \in \mathcal{R}_{A_\nu}(t_1)$ and $x \in X$ with $\|x\| \geq R_0 + 2|t_1 - t_2|$. By (5.11), there exists $y \in \gamma(X) \setminus B(0, R_0)$ so that $d(x, y) < 2|t_1 - t_2|$. Let I denote the interval with endpoints x and y . If $y \notin \text{Bd}(A_\nu)$, then $I \cap \text{Bd}(A_\nu) \neq \emptyset$, since $\text{Bd}(A_\nu)$ separates between x and y . Thus there exists $y_1 \in I \cap \text{Bd}(A_\nu)$ so that $y_1 \in f^{-1}(t_2)$, $\|y_1\| \geq R_0$ and $d(y, y_1) < 2|t_1 - t_2|$. Thus, by (5.9), $y_1 \in \gamma(X)$ and $\gamma(X) \cap \text{Bd}(A_\nu) \neq \emptyset$. Hence, by (5.14), $\gamma(X) \subset \text{Bd}(A_\nu)$ and (5.17) holds.

Since γ is one-to-one, (5.17) immediately implies that

$$(5.18) \quad \text{card}(\mathcal{R}_{A_\nu}(t_1)) \leq \#e(\text{Bd}(A_\nu)).$$

Since sets $\{A_\nu\}_{\nu=1}^m$ are disjoint, by (5.7) and by part (b) applied to t_2 and the set $A = \mathbb{R}^2$, we get

$$\begin{aligned} \#e(f^{-1}(t_1)) &= \text{card}(\mathcal{R}(t_1)) = \sum_{\nu=1}^m \text{card}(\mathcal{R}_{A_\nu}(t_1)) \\ &\leq \sum_{\nu=1}^m \#e(\text{Bd}(A_\nu)) = \#e(f^{-1}(t_2)). \end{aligned}$$

Since, by part (a), $\#e(f^{-1}(t_1)) = \#e(f^{-1}(t_2))$, we conclude that, for all $\nu \in \{1, \dots, m\}$,

$$\#e(f^{-1}(t_1) \cap A_\nu) = \text{card}(\mathcal{R}_{A_\nu}(t_1)) = \#e(\text{Bd}(A_\nu)),$$

which ends the proof of part (c). □

For the proof of the main theorem we will need one more lemma.

Lemma 5.13. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and A be a component of $f^{-1}(t, \infty)$. Then:*

(a) *for every $s > t$,*

$$\#c(f^{-1}(s) \cap A) \leq \#c(f^{-1}(t) \cap \text{Bd}(A));$$

(b) *there exists $\varepsilon > 0$ so that for every $s \in (t, t + \varepsilon)$,*

$$\#c(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \text{Bd}(A));$$

(c) *let $t_1 > t$ and let $\{C_\nu\}_{\nu=1}^k$ be components of $f^{-1}(t_1, \infty)$ which are contained in A , then*

$$\sum_{\nu=1}^k \#c(f^{-1}(t_1) \cap \text{Bd}(C_\nu)) = \#c(f^{-1}(t) \cap \text{Bd}(A));$$

(d) *let $m = \#c(f^{-1}(t) \cap \text{Bd}(A))$, then there exists a set $T_A \subset (t, \infty)$ with $\text{card}(T_A) \leq m - 1$, so that for every $s \in (t, \infty) \setminus T_A$,*

$$\#c(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \text{Bd}(A)).$$

Proof. Let $s > t$. By Lemma 5.12(b) and (c) and by Corollary 5.5,

$$\#e(f^{-1}(s) \cap A) = \#e(\text{Bd}(A)) = 2\#c(f^{-1}(t) \cap \text{Bd}(A)).$$

Since, by Proposition 5.4, each component of $f^{-1}(s) \cap A$ has at least 2 ends, we get that

$$\#c(f^{-1}(s) \cap A) \leq \frac{1}{2}\#e(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \text{Bd}(A)),$$

and part (a) is proven.

For part (b) we assume, without loss of generality, that the co-Lipschitz constant of f is equal to 1. Denote

$$l \stackrel{\text{def}}{=} \#c(f^{-1}(t) \cap \text{Bd}(A)),$$

and let $\{L_j\}_{j=1}^l$ be the components of $f^{-1}(t) \cap \text{Bd}(A)$.

By Proposition 5.4, the intersection of any component of $f^{-1}(t)$ with $\text{Bd}(A)$ is connected and therefore each L_j is contained in a different component of $f^{-1}(t)$. Hence, by Corollary 5.9, there exists $\delta > 0$ so that, for all $i, j \in \{1, \dots, l\}, i \neq j$,

$$d(L_i, L_j) \geq \delta.$$

Define for $j \in \{1, \dots, l\}$,

$$U_j \stackrel{\text{def}}{=} \bigcup_{x \in L_j} B(x, \frac{\delta}{2}) \cap A.$$

Then $\{U_j\}_{j=1}^l$ are connected, mutually disjoint, open subsets of A .

We note that, for any $y \in \mathbb{R}^2$,

$$(5.19) \quad d(y, f^{-1}(t)) \geq \frac{\delta}{2} \implies |f(y) - t| \geq \frac{\delta}{2}.$$

Indeed, if $d(y, f^{-1}(t)) \geq \frac{\delta}{2}$, then $t \notin f(B(y, \frac{\delta}{2}))$. But, since f is co-Lipschitz with constant 1, $f(B(y, \frac{\delta}{2})) \supset B(f(y), \frac{\delta}{2})$. Thus $t \notin B(f(y), \frac{\delta}{2})$ and (5.19) holds. Hence, for all $s \in (t, t + \frac{\delta}{2})$,

$$(5.20) \quad f^{-1}(s) \cap A \subset \bigcup_{j=1}^l U_j.$$

Now fix $j_0 \in \{1, \dots, l\}$, and let $x \in L_{j_0}$ and $y \in \text{Bd}(U_{j_0}) \setminus \text{Bd}(A)$. Then $d(y, f^{-1}(t)) \geq \frac{\delta}{2}$.

Moreover, since L_{j_0} is locally connected and U_{j_0} is open, $\overline{U_{j_0}}$ is arcwise connected and there exists a continuous function $\sigma : [0, 1] \rightarrow \overline{U_{j_0}}$ so that $\sigma(0) = x, \sigma(1) = y$ and $\sigma(\lambda) \in U_{j_0}$ for $\lambda \in (0, 1)$. Define $g : [0, 1] \rightarrow \mathbb{R}$ as $g = f \circ \sigma$. Then

$$\begin{aligned} g(0) &= f(x) = t, \\ g(1) &= f(y) \geq t + \frac{\delta}{2}, \quad \text{by (5.19)}. \end{aligned}$$

By the Intermediate Value Theorem, for every $s \in (t, t + \frac{\delta}{2})$, there exists at least one $\lambda_s \in (0, 1)$ so that

$$s = g(\lambda_s) = f(\sigma(\lambda_s)).$$

Since $\sigma(\lambda_s) \in U_{j_0}$, we conclude that, for every $j_0 \in \{1, \dots, l\}$ and every $s \in (t, t + \frac{\delta}{2})$,

$$f^{-1}(s) \cap U_{j_0} \neq \emptyset.$$

Since sets $\{U_j\}_{j=1}^l$ are mutually disjoint and by (5.20), we get that for all $s \in (t, t + \frac{\delta}{2})$,

$$\#c(f^{-1}(s) \cap A) \geq l = \#c(f^{-1}(t) \cap \text{Bd}(A)).$$

This, together with part (a), concludes the proof of part (b).

Part (c) follows by the following computation:

$$\begin{aligned} \sum_{\nu=1}^k \#c(f^{-1}(t_1) \cap \text{Bd}(C_\nu)) &= \frac{1}{2} \sum_{\nu=1}^k \#e(\text{Bd}(C_\nu)), && \text{by Corollary 5.5,} \\ &= \frac{1}{2} \#e(f^{-1}(t_1) \cap A), && \text{by Lemma 5.12(b),} \\ &= \frac{1}{2} \#e(\text{Bd}(A)), && \text{by Lemma 5.12(c),} \\ &= \#c(f^{-1}(t) \cap \text{Bd}(A)), && \text{by Corollary 5.5.} \end{aligned}$$

To prove part (d), we proceed inductively with respect to m .

If $m = 1$, then by part (a), for every $s \in (t, \infty)$,

$$\#c(f^{-1}(s) \cap A) \leq 1,$$

and since $f(A) = (t, \infty)$, we have $f^{-1}(s) \cap A \neq \emptyset$, and hence $\#c(f^{-1}(s) \cap A) \geq 1$. Therefore part (d) holds with $T_A = \emptyset$, as desired.

For the induction, we assume that part (d) holds for all $m < m_0$, where $m_0 \geq 2$.

Now suppose that

$$\#c(f^{-1}(t) \cap \text{Bd}(A)) = m_0 \geq 2.$$

Define

$$t_1 \stackrel{\text{def}}{=} \sup\{\tau \in (t, \infty) : \forall s \in (t, \tau) \#c(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \text{Bd}(A))\}.$$

If $t_1 = \infty$ there is nothing to prove, so suppose that $t_1 < \infty$. By part (b), $t_1 > t$ and

$$\#c(f^{-1}(t_1) \cap A) \neq \#c(f^{-1}(t) \cap \text{Bd}(A)).$$

By part (a), this implies that

$$(5.21) \quad \#c(f^{-1}(t_1) \cap A) < \#c(f^{-1}(t) \cap \text{Bd}(A)).$$

Let $\{C_\nu\}_{\nu=1}^k$ denote all components of $f^{-1}(t_1, \infty) \cap A$. Then, by (5.21), for each $\nu \leq k$,

$$(5.22) \quad \begin{aligned} \#c(f^{-1}(t_1) \cap \text{Bd}(C_\nu)) &\leq \#c(f^{-1}(t_1) \cap A) \\ &< \#c(f^{-1}(t) \cap \text{Bd}(A)) = m_0. \end{aligned}$$

Hence, by the inductive hypothesis, for each $\nu \leq k$ there exists a set $T_\nu = T_{C_\nu} \subset (t_1, \infty)$ with $\text{card}(T_\nu) \leq \#c(f^{-1}(t_1) \cap \text{Bd}(C_\nu)) - 1$, so that for every $s \in (t_1, \infty) \setminus T_\nu$,

$$(5.23) \quad \#(f^{-1}(s) \cap C_\nu) = \#c(f^{-1}(t_1) \cap \text{Bd}(C_\nu)).$$

Set

$$T_A = \bigcup_{\nu=1}^k T_\nu \cup \{t_1\}.$$

Then for every $s \in (t_1, \infty) \setminus T_A$, we have:

$$\begin{aligned} \#c(f^{-1}(s) \cap A) &= \sum_{\nu=1}^k \#c(f^{-1}(s) \cap C_\nu) \\ &= \sum_{\nu=1}^k \#c(f^{-1}(t_1) \cap \text{Bd}(C_\nu)), && \text{by (5.23),} \\ &= \#c(f^{-1}(t) \cap \text{Bd}(A)), && \text{by part (c).} \end{aligned}$$

To finish the proof we only need to estimate the cardinality of the set T_A . We have

$$\begin{aligned} \text{card}(T_A) &\leq \sum_{\nu=1}^k \text{card}(T_\nu) + 1 \\ &\leq \sum_{\nu=1}^k [\#c(f^{-1}(t_1) \cap \text{Bd}(C_\nu)) - 1] + 1 \\ &= \#c(f^{-1}(t) \cap \text{Bd}(A)) + (1 - k), && \text{by part (c).} \\ &\leq \#c(f^{-1}(t) \cap \text{Bd}(A)) - 1, && \text{since, by part (c) and (5.22), } k \geq 2, \end{aligned}$$

which ends the proof of part (d). \square

We are now ready for the proof of our main theorem.

Proof of Theorem 5.1. To prove part (1) we note that by Corollary 5.7, number of ends of any level set $f^{-1}(t)$, for $t \in \mathbb{R}$, is finite and bounded by a constant M depending only on the co-Lipschitz constant of f and its modulus of uniform continuity.

By Lemma 5.12(a), $\#e(f^{-1}(t))$ does not depend on the value of $t \in \mathbb{R}$. To see that $\#e(f^{-1}(t))$ is even, let $\{A_\lambda(t)\}_{\lambda=1}^l$ be the components of $\mathbb{R}^2 \setminus f^{-1}(t)$. Then we have

$$\begin{aligned} \#e(f^{-1}(t)) &= \sum_{\lambda=1}^l \#e(\text{Bd}(A_\lambda(t))), && \text{by Lemma 5.12(b),} \\ &= 2 \sum_{\lambda=1}^l \#c(f^{-1}(t) \cap \text{Bd}(A_\lambda(t))), && \text{by Corollary 5.5.} \end{aligned}$$

Thus $\#e(f^{-1}(t))$ is even.

The moreover statement follows from Corollary 5.10, and hence part (1) is proven.

For the proof of part (2), let t_0 be any real number, say $t_0 = 0$, and let $\{A_\nu\}_{\nu=1}^l$ be all the components of $f^{-1}(0, \infty)$. By Lemma 5.13(d), for every $\nu \leq l$, there exists a set $T_{A_\nu} \subset (0, \infty)$ with $\text{card}(T_{A_\nu}) \leq \#c(f^{-1}(0) \cap \text{Bd}(A_\nu)) - 1$, so that for all $s \in (0, \infty) \setminus T_{A_\nu}$,

$$(5.24) \quad \#c(f^{-1}(s) \cap A_\nu) = \#c(f^{-1}(0) \cap \text{Bd}(A_\nu)).$$

Define

$$T_+^0 = \bigcup_{\nu=1}^l T_{A_\nu}.$$

Note that

$$(5.25) \quad \begin{aligned} \sum_{\nu=1}^l \#c(f^{-1}(0) \cap \text{Bd}(A_\nu)) &= \frac{1}{2} \sum_{\nu=1}^l \#e(\text{Bd}(A_\nu)), && \text{by Corollary 5.5,} \\ &= \frac{1}{2} \#e(f^{-1}(0)), && \text{by Lemma 5.12(b).} \end{aligned}$$

Therefore for every $s \in (0, \infty) \setminus T_+^0$, we have:

$$\begin{aligned} \#c(f^{-1}(s)) &= \sum_{\nu=1}^l \#c(f^{-1}(s) \cap A_\nu) \\ &= \sum_{\nu=1}^l \#c(f^{-1}(0) \cap \text{Bd}(A_\nu)), && \text{by (5.24),} \\ &= \frac{1}{2} \#e(f^{-1}(0)), && \text{by (5.25),} \\ &= n, && \text{by part (1).} \end{aligned}$$

Similarly

$$\begin{aligned} \text{card}(T_+^0) &\leq \sum_{\nu=1}^l \text{card}(T_{A_\nu}) \leq \sum_{\nu=1}^l [\#c(f^{-1}(0) \cap \text{Bd}(A_\nu)) - 1] \\ &= \frac{1}{2} \#e(f^{-1}(0)) - l, && \text{by (5.25),} \\ &\leq n - 1, && \text{by part (1).} \end{aligned}$$

Next, we note that in an identical way (e.g. by replacing function f by $-f$) one can define a set $T_-^0 \subset (-\infty, 0)$ with $\text{card}(T_-^0) \leq n - 1$, so that for every $s \in (-\infty, 0) \setminus T_-^0$, $\#c(f^{-1}(s)) = n$. Define

$$T_f^0 = T_+^0 \cup T_-^0 \cup \{0\}.$$

Clearly, $\text{card}(T_f^0) \leq 2n - 1$. Now, let $t_{00} \in \mathbb{R}$ be such that $t_{00} < t$ for every $t \in T_f^0$. Then we clearly have $\#c(f^{-1}(s)) = n$, for every $s \leq t_{00}$. Same way as was done above for $t_0 = 0$, we construct a set $T_+^{00} \subset (t_{00}, \infty)$ so that for every $s \in (t_{00}, \infty) \setminus T_+^{00}$, we have $\#c(f^{-1}(s)) = n$ and $\text{card}(T_+^{00}) \leq n - 1$. Then part (2a) holds for $T_f \stackrel{\text{def}}{=} T_+^{00}$.

Part (2b) follows immediately by Corollary 5.6.

For part (2c), note that by Proposition 5.4, each component of $f^{-1}(t)$ has at least 2 ends. Since by parts (1) and (2a) for all $t \in \mathbb{R} \setminus T_f$, $f^{-1}(t)$ has n components and $2n$ ends, we conclude that each component of $f^{-1}(t)$ has exactly 2 ends. Hence, by Proposition 5.4,

each component K of $f^{-1}(t)$ separates the plane into exactly 2 components. Further, by Theorem 4.11, K has a representation of the form

$$K = K_0 \cup K_1 \cup K_2,$$

where $K_1 \cup K_2 = \emptyset$, and for $i = 1, 2$, $K_i \cap K_0$ consists of exactly one point which is on endpoint of both K_i and K_0 , K_1 and K_2 are both homeomorphic with $[0, \infty)$, and K_0 is a compact connected tree with exactly 2 endpoints. Thus K_0 is homeomorphic with $[0, 1]$ (either by the construction of K_0 described in the proof of Theorem 4.11, or by the classical characterization of the interval as a continuum with exactly 2 non-cut points cf. e.g. [23, Theorem III.(6.2)]). Therefore K is homeomorphic with $(-\infty, 0] \cup [0, 1] \cup [1, \infty) = (-\infty, \infty)$, which ends the proof of part (2c).

For part (3a) we note that for all $t_i \in T_f$, $\#c(f^{-1}(t_i)) \neq n$. Since $\#e(f^{-1}(t_i)) = 2n$ by part (1), and each component has at least 2 ends, by Proposition 5.4, we conclude that $\#c(f^{-1}(t_i)) < n$, i.e. part (3a) holds. Parts (3b) and (3c) follow immediately from Corollary 5.6 and Theorem 4.11, respectively. \square

6. EXAMPLE

In this section we present an example, mentioned in the Introduction, of a uniform quotient map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with a non-locally connected level set.

Example 6.1. *There exists a uniform quotient map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that $f^{-1}(0)$ is not locally connected.*

Construction. Let $z_n = (\frac{1}{n}, (-1)^n) \in \mathbb{R}^2$ for $n \in \mathbb{Z} \setminus \{0\}$, and let I_n be a segment in \mathbb{R}^2 with endpoints z_n, z_{n+1} , when $n > 0$, or z_n, z_{n-1} when $n < 0$. Let I_0 be the vertical segment with endpoints $(0, 1)$ and $(0, -1)$, and let I_+, I_- be the following two half-lines:

$$I_+ = \{(x, -1) : x \geq 1\}, \quad I_- = \{(x, -1) : x \leq -1\}$$

Define K to be the sum of all these segments

$$K \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{Z}} I_n \cup I_+ \cup I_-$$

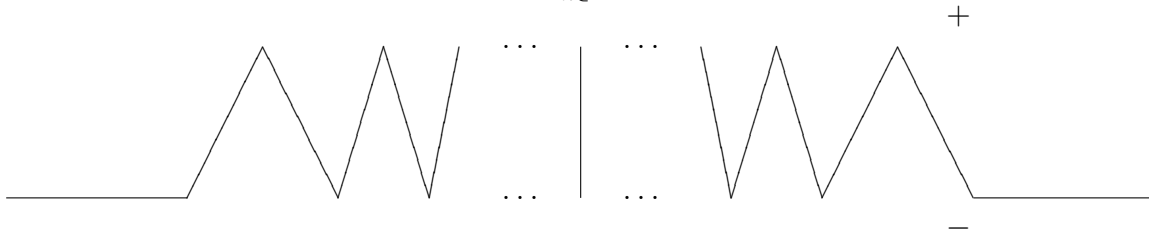


FIGURE 6.1. Set K .

Set K is connected but not locally connected and it separates the plane into two regions. We define the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ as the distance from K multiplied in each component of K by the sign indicated. Then, clearly, $K = f^{-1}(0)$. It is also clear that f is Lipschitz. We will show that f is co-uniformly continuous with

$$(6.1) \quad \omega(r) = \begin{cases} \frac{r^3}{16000} & \text{if } r < \frac{1}{10}, \\ \frac{1}{16 \cdot 10^6} & \text{if } r \geq \frac{1}{10}. \end{cases}$$

We achieve this in a number of steps.

Step 1. Let $A_n = (\frac{1}{n}, 0)$, $B_n = (\frac{1}{n}, (-1)^n)$, $C_n = (\frac{1}{n-1}, (-1)^{n-1})$, $D_n = (\frac{1}{n+1}, (-1)^{n+1})$ for $n \in \mathbb{Z} \setminus \{0, 1, -1\}$. Let $\alpha_n = \sphericalangle A_n B_n C_n$ and $\beta_n = \sphericalangle A_n B_n D_n$, where both angles are assumed to be positive. Then

$$\frac{1}{2|n|(|n| - 1)} \geq \sin \alpha_n \geq \frac{1}{3|n|(|n| - 1)}$$

$$\frac{1}{2|n|(|n| + 1)} \geq \sin \beta_n \geq \frac{1}{3|n|(|n| + 1)}.$$

Proof of Step 1. Without loss of generality, we assume that $n > 1$. We illustrate α_n and β_n on Figure 6.2.

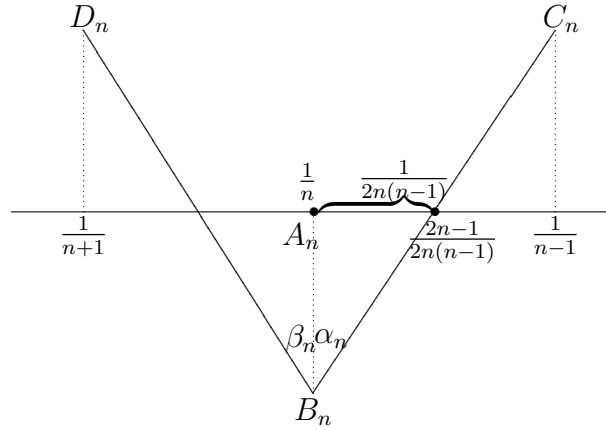


FIGURE 6.2.

It is not difficult to compute that, as indicated on Figure 6.2,

$$\sin \alpha_n = \frac{\frac{1}{2n(n-1)}}{\sqrt{1 + (\frac{1}{2n(n-1)})^2}} = \frac{1}{\sqrt{4n^2(n-1)^2 + 1}}.$$

Thus,

$$\frac{1}{2n(n-1)} = \frac{1}{\sqrt{4n^2(n-1)^2}} \geq \sin \alpha_n \geq \frac{1}{\sqrt{9n^2(n-1)^2}} = \frac{1}{3n(n-1)}.$$

Similarly,

$$\sin \beta_n = \frac{\frac{1}{2n(n+1)}}{\sqrt{1 + (\frac{1}{2n(n+1)})^2}} = \frac{1}{\sqrt{4n^2(n+1)^2 + 1}}.$$

Thus

$$\frac{1}{2n(n+1)} = \frac{1}{\sqrt{4n^2(n+1)^2}} \geq \sin \beta_n \geq \frac{1}{\sqrt{9n^2(n+1)^2}} = \frac{1}{3n(n+1)}.$$

□

Step 2. If $x = (x_1, x_2) \in K$ and $x_1 = 0$ then for $r \leq \frac{1}{10}$

$$(6.2) \quad f(B(x, r)) \supset B(f(x), \frac{r^3}{2000}).$$

Proof of Step 2. We first consider the case when $|x_2| \leq 1$, as illustrated on Figure 6.3.

Let n be the smallest odd number so that

$$(6.3) \quad \frac{1}{n-1} \leq \frac{r}{32}.$$

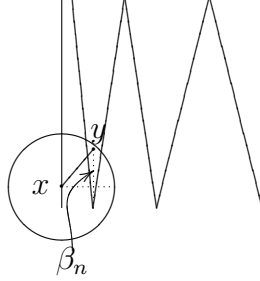


FIGURE 6.3.

Then there exists $y = (y_1, y_2) \in B(x, r)$ so that $y_1 = \frac{1}{n}$ and $y_2 \geq x_2 + \frac{r}{2}$. Then

$$f(y) = d(y, K) \geq \frac{r}{2} \sin \beta_n \geq \frac{r}{2} \cdot \frac{1}{3n(n+1)}.$$

By (6.3) and since $r \leq \frac{1}{10}$ we see that $n > 4$ and $\frac{1}{n-3} > \frac{r}{2}$. Thus

$$\begin{aligned} \frac{1}{n} &= \frac{n-3}{n} \cdot \frac{1}{n-3} \geq \frac{1}{4} \cdot \frac{r}{2}, \\ \frac{1}{n+1} &= \frac{n-3}{n+1} \cdot \frac{1}{n-3} \geq \frac{1}{5} \cdot \frac{r}{2}. \end{aligned}$$

Hence

$$f(y) = d(y, k) \geq \frac{r}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{r}{2} \cdot \frac{1}{5} \cdot \frac{r}{2} = \frac{r^3}{480}.$$

Similarly there exists $z = (z_1, z_2) \in B(x, r)$ so that $z_1 = \frac{1}{n+1}$ and $z_2 \leq x_2 - \frac{r}{2}$. Then $f(z) = -d(z, K)$ and

$$d(z, K) \geq \frac{r}{2} \sin \beta_{n+1} \geq \frac{r}{2} \cdot \frac{1}{3(n+1)(n+2)} \geq \frac{r}{2} \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{r}{2} \cdot \frac{1}{6} \cdot \frac{r}{2} = \frac{r^3}{720}.$$

Thus (6.2) is satisfied. \square

Step 3. If $x = (x_1, x_2) \in K$ and $x_1 \neq 0$ then for $r \leq \frac{1}{10}$, (6.2) is satisfied.

Proof of Step 3. Since $x_1 \neq 0$, thus there exists $m \in \mathbb{Z}$ so that x belongs to I_m , the segment which connects points $(\frac{1}{m}, (-1)^m)$ and $(\frac{1}{m+1}, (-1)^{m+1})$. Without loss of generality we will assume that $m > 0$. Let n denote the smallest odd number so that

$$\frac{1}{n-1} \leq \frac{r}{2}.$$

If $n < m-1$ we proceed in a way very similar to Step 2. Since $x_1 > 0$, we see that there exists $y = (y_1, y_2) \in B(x, r)$ and $z = (z_1, z_2) \in B(x, r)$ so that $y_1 = \frac{1}{n}$, $y_2 \geq x_2 + \frac{r}{2}$, $z_1 = \frac{1}{n+1}$, $z_2 \leq x_2 - \frac{r}{2}$. Then $f(y) = d(y, K)$ and $f(z) = -d(z, K)$. Further, similarly as in Step 2,

$$\begin{aligned} d(y, K) &\geq \frac{r^3}{480}, \\ d(z, K) &\geq \frac{r^3}{720}. \end{aligned}$$

Thus (6.2) is satisfied.

If $n \geq m - 1$ and $m > 3$ (the case when $m \leq 3$ is done similarly and we leave the details to the interested reader) then

$$(6.4) \quad \frac{1}{m-3} > \frac{r}{2}.$$

Now let $t \in [0, 1]$ be such that $x = t(\frac{1}{m}, (-1)^m) + (1-t)(\frac{1}{m+1}, (-1)^{m+1})$, see Figure 6.4.

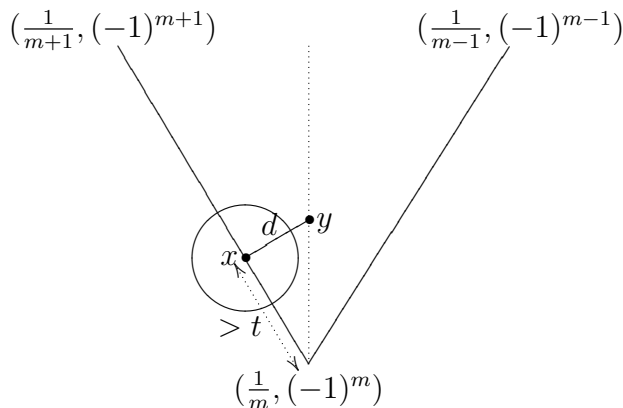


FIGURE 6.4.

Let $y = (y_1, y_2)$ be the point with $y_1 = \frac{1}{m}$, so that the segment $[x, y]$ with endpoints x and y is perpendicular to I_m . If $t \geq \frac{r}{3}$ then, by (6.4), (since $m > 3$),

$$d = d(x, y) \geq t \cdot \sin \beta_m \geq \frac{r}{3} \cdot \frac{1}{3m(m+1)} \geq \frac{r}{3} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{r}{2} \cdot \frac{1}{5} \cdot \frac{r}{2} = \frac{r^3}{720}.$$

If $d < r$ then $y \in B(x, r)$ and $f(y) = d \geq \frac{r^3}{720}$. If $d \geq r$, let y_r denote a point in the segment with endpoints x and y so that $d(x, y_r) = r$. Then $f(y_r) = r$ and $f(B(x, r)) \supset f([x, y_r]) \supset [0, r)$.

Next we consider the case when $t < \frac{r}{3}$, as illustrated on Figure 6.5

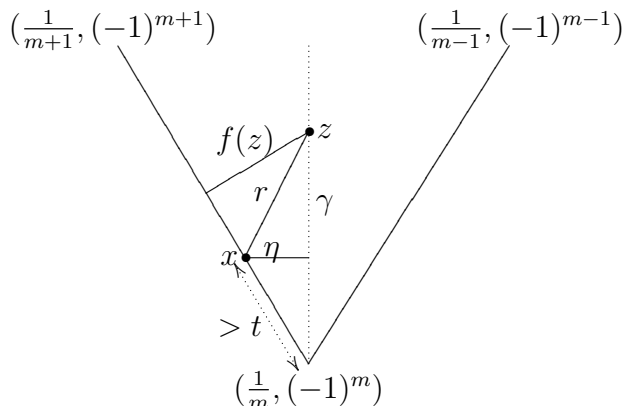


FIGURE 6.5.

Then

$$\eta = d(x, \{v = (v_1, v_2) : v_1 = \frac{1}{m}\}) \leq 2t \sin \beta_m \leq 2 \cdot \frac{r}{3} \cdot \frac{1}{m(m+1)} \leq \frac{r}{3}.$$

Hence

$$\gamma = \sqrt{r^2 - \eta^2} \geq \sqrt{r^2 - \frac{r^2}{9}} = r \frac{\sqrt{8}}{3}.$$

Thus there exists $z \in B(x, r)$ with $z_1 = \frac{1}{m}$ and $z_2 \geq x_2 + r \frac{\sqrt{8}}{3} \geq -1 + \frac{2}{3}r$. We have

$$f(z) = d(z, K) \geq \frac{2}{3}r \cdot \sin \beta_m \geq \frac{2}{3}r \cdot \frac{1}{3m(m+1)} \geq \frac{2}{3}r \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{r}{2} \cdot \frac{1}{5} \cdot \frac{r}{2} = \frac{r^3}{360}.$$

Thus for all $t \in [0, 1]$ we conclude that $f(B(x, r)) \supset [0, \frac{r^3}{720}]$.

A similar computation shows that $f(B(x, r))$ contains also a sufficiently large negative interval, so that (6.2) holds. \square

Step 4. If $d(x, K) = d > 0$ then

(a) if $f(x) > 0$ then

$$(6.5) \quad f(B(x, r)) \supset (\max(d - r, 0), d + \frac{r^3}{480});$$

(b) if $f(x) < 0$ then

$$(6.6) \quad f(B(x, r)) \supset (-d - \frac{r^3}{480}, \min(d + r, 0)).$$

Proof of Step 4. We will assume without loss of generality that $f(x) > 0$. The case when $f(x) < 0$ is proven identically. To prove (6.5) we will consider two cases. First we assume that $x = (x_1, x_2)$ where $|x_1|, |x_2| \leq 1$. Since $x_1 \neq 0$, this implies that there exists $m \in \mathbb{Z}$ (say, $m > 0$) so that x lies inside the triangle with vertices $(\frac{1}{m-1}, (-1)^{m-1})$, $(\frac{1}{m}, (-1)^m)$, $(\frac{1}{m+1}, (-1)^{m+1})$, as illustrated on Figure 6.6

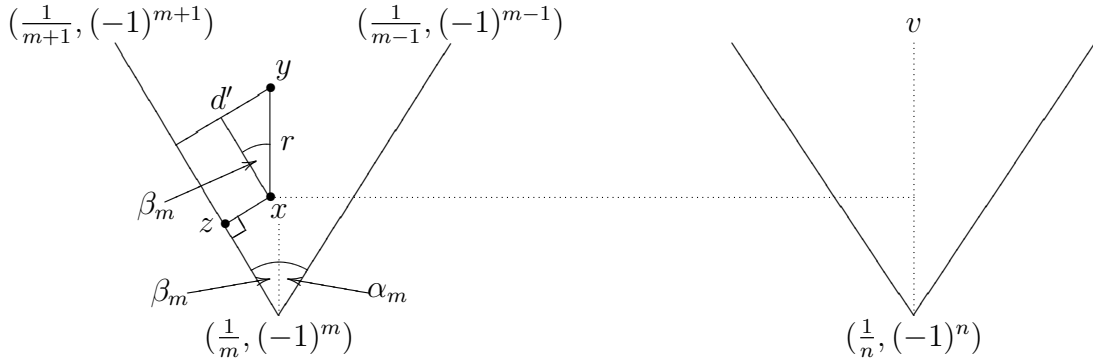


FIGURE 6.6.

If $\frac{1}{m+1} \geq \frac{r}{4}$ we consider a point $y = (y_1, y_2) \in \overline{B(x, r)}$ so that $y_1 = x_1$, $y_2 = x_2 + r$. Then

$$\begin{aligned} f(y) = d' &= d(y, K) \geq d + r \sin \beta_m \\ &\geq d + r \frac{1}{3m(m+1)} \geq d + r \cdot \frac{1}{3} \cdot \frac{r}{4} \cdot \frac{r}{4} \\ &= d + \frac{r^3}{48}. \end{aligned}$$

If $\frac{1}{m+1} < \frac{r}{4}$ then we proceed similarly to Steps 2 and 3. Let n be the smallest odd number so that

$$\frac{1}{n-1} > \frac{r}{2}$$

then (6.7) is satisfied. Equation (6.8) is true when $r \leq 2d$. Thus, next we assume that

$$\frac{1}{10} \geq r \geq 2d.$$

We will also assume, without loss of generality, that $f(x) > 0$. Let $y \in K$ be such that $d(x, y) = d$. Then $B(x, r) \supset B(y, r - d)$. By Steps 2 and 3, (6.2) holds and we have

$$f(B(y, r - d)) \supset \left(-\frac{(r - d)^3}{2000}, 0\right].$$

Note that

$$\frac{(r - d)^3}{2000} \geq \frac{(\frac{1}{2}r)^3}{2000} = \frac{r^3}{16000}.$$

Thus

$$f(B(x, r)) \supset \left(-\frac{r^3}{16000}, d\right].$$

On the other hand, by Step 4,

$$f(B(x, r)) \supset \left(0, d + \frac{r^3}{480}\right).$$

Thus (6.7) is satisfied. □

This ends the proof that f is co-uniformly continuous with the modulus ω defined in (6.1). □

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