LEVEL SETS OF UNIFORM QUOTIENT MAPPINGS FROM $\mathbb{R}^n$ TO $\mathbb{R}$ DO NOT NEED TO BE LOCALLY CONNECTED

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Abstract. We give an example of a uniform quotient map from $\mathbb{R}^2$ to $\mathbb{R}$ which has non-locally connected level sets.

1. Introduction

Let X, Y be metric spaces. A mapping $f : X \longrightarrow Y$ is said to be a *uniform quotient mapping* if there exist functions $\omega, \Omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(r) > 0$ for all $r > 0$ and $\lim_{r\to 0} \Omega(r) = 0$ so that for all $x \in X$ and all $r > 0$:

$$B(f(x), \omega(r)) \subset f(B(x, r)) \subset B(f(x), \Omega(r)), \tag{1.1}$$

where $B(x, r)$ denotes the open ball with center x and radius r.

Notice that the right hand inclusion means that f is uniformly continuous. The mapping f is called *co-uniformly continuous* if the left hand inclusion in (**??**) is satisfied. There is no restriction in assuming that the functions ω and Ω are continuous and increasing. They are called *moduli of co-uniform and uniform continuity of f*, respectively. If the functions ω and Ω are linear, i.e. if there exist constants $c, L > 0$ so that for all $x \in X$ and all $r > 0$:

$$B(f(x), cr) \subset f(B(x, r)) \subset B(f(x), Lr), \tag{1.2}$$

then f is called a *Lipschitz quotient mapping.* Clearly the right hand inclusion in (**??**) means that f is a Lipschitz mapping. If f satisfies the left hand inclusion of (**??**), f is called a *co-Lipschitz mapping.* Constants c and L are called *co-Lipschitz and Lipschitz constants of f*, respectively. The study of uniform and Lipschitz quotient mappings was initiated in [**?**], see also [**?**] for the comprehensive introduction of the subject. The structure of Lipschitz and uniform quotient mappings $f : X \longrightarrow Y$, when X and Y are finite dimensional was studied by Johnson, Lindenstrauss, Preiss and Schechtman in [**?**]. They obtained most complete results for the case of $X = Y = \mathbb{R}^2$ and they posed some questions about the structure of level sets of uniform and Lipschitz quotient mappings $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. In answer to these questions in [**?**] we proved that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n \geq 2$, is a uniform quotient mapping then for every $t \in \mathbb{R}$, $f^{-1}(t)$ has a bounded number of components, each component of $f^{-1}(t)$ separates $\mathbb{R}^n$ and the upper bound of the number of components depends only on n and the moduli of co-uniform and uniform continuity of f [**?**, Theorem 2.4]. We also proved that if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a co-Lipschitz uniformly continuous mapping then the number of components of $f^{-1}(t)$ and of $\mathbb{R}^2 \setminus f^{-1}(t)$ is constant for all $t \in \mathbb{R} \setminus T_f$, where T_f is a finite subset of $\mathbb{R}$. Moreover, for all $t \in \mathbb{R}^2 \setminus f^{-1}(t)$, the level set $f^{-1}(t)$ is a finite disjoint union of homeomorphic copies of $\mathbb{R}$ and for $t \in T_f$, each component of $f^{-1}(t)$ has a finite tree structure [**?**, Theorem 5.1]. Maleva [**?**] refined the conclusions of [**?**, Theorem 5.1] by providing some additional information about the geometric structure of level sets of co-Lipschitz uniformly continuous mappings from $\mathbb{R}^2$

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to $\mathbb{R}$, e.g. it follows from [?] that no component of $f^{-1}(t)$ can contain a parabola. Maleva [?] also gave exact estimates for the allowable number of components of $f^{-1}(t)$ depending on the co-Lipschitz constant and modulus of uniform continuity of f only.

Our method of proof of [?, Theorem 5.1] depended on a careful analysis of topological properties of level sets $f^{-1}(t)$, their end points and their structure at infinity. The crucial property that we used in a very essential way is the fact that level sets $f^{-1}(t)$ are locally connected when f is a co-Lipschitz uniformly continuous map from $\mathbb{R}^2$ to $\mathbb{R}$ [?, Proposition 3.5].

In [?] we posed a question whether level sets of co-Lipschitz uniformly continuous maps or of Lipschitz quotient maps from $\mathbb{R}^n$ to $\mathbb{R}$ are locally connected when $n > 2$. This question has been recently answered affirmatively by Terpai [?]. However, as we announced without proof in [?], there exist uniform quotient maps from $\mathbb{R}^2$ to $\mathbb{R}$ with non-locally connected level sets. In the present note we prove this claim.

Throughout the paper we use standard notation, as may be found in [?, ?, ?, ?].

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2. Example

Recall that a topological space S is said to be *locally connected at a point* x if for every open set U containing x there is a connected open set V so that $x \in V \subset U$. The space S is *locally connected* if it is locally connected at each point.

Example 2.1. *Let* $z_n = (\frac{1}{n}, (-1)^n) \in \mathbb{R}^2$ *for* $n \in \mathbb{Z} \setminus \{0\}$*, and let* I_n *be a segment in* $\mathbb{R}^2$ *with endpoints* $z_n,\ z_{n+1}$*, when* $n > 0$*, or* $z_n,\ z_{n-1}$ *when* $n < 0$*. Let* I_0 *be the vertical segment with endpoints* $(0, 1)$ *and* $(0, -1)$*, and let* $I_+ = \{(x, -1) : x \geq 1\}$*,* $I_- = \{(x, -1) : x \leq -1\}$ *be two half-lines. Define* K *to be the union of all these segments* $K \overset{def}{=} \bigcup_{n \in \mathbb{Z}} I_n \cup I_+ \cup I_-$*.*

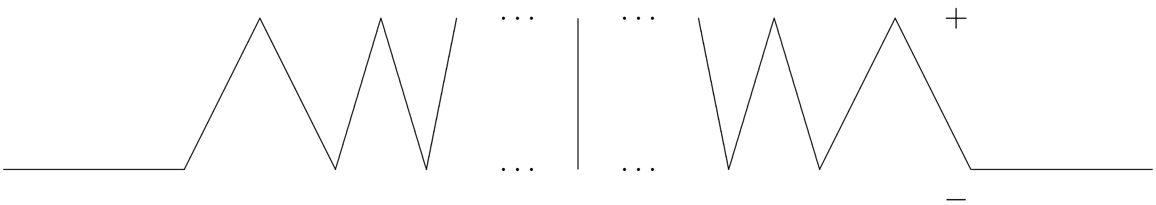

Figure **2.1.** Set K.

The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is defined as the distance from* K *multiplied in each component of* $\mathbb{R}^2 \setminus K$ *by the sign indicated. Then* f *is Lipschitz and co-uniformly continuous with modulus*

$$\omega(r) = \begin{cases} \frac{r^3}{16000} & \text{if } r < \frac{1}{10}, \\ \frac{1}{16 \cdot 10^6} & \text{if } r \geq \frac{1}{10}. \end{cases}$$

Moreover $f^{-1}(0) = K$*, which is connected but not locally connected.*

Proof. It is clear that f is Lipschitz. To show that f is co-uniformly continuous with modulus ω we need a number of steps.

Step 1. *Let* $A_n = (\frac{1}{n}, 0)$, $B_n = (\frac{1}{n}, (-1)^n)$, $C_n = (\frac{1}{n-1}, (-1)^{n-1})$, $D_n = (\frac{1}{n+1}, (-1)^{n+1})$ *for* $n \in \mathbb{Z} \setminus \{0, 1, -1\}$. *Let* $\alpha_n = \sphericalangle A_n B_n C_n$ *and* $\beta_n = \sphericalangle A_n B_n D_n$, *where both angles are assumed to be positive. Then*

$$\frac{1}{2|n|(|n|-1)} \geq \sin\alpha_n \geq \frac{1}{3|n|(|n|-1)},$$
$$\frac{1}{2|n|(|n|+1)} \geq \sin\beta_n \geq \frac{1}{3|n|(|n|+1)}.$$

Moreover $\alpha_n < \alpha_m$ *and* $\beta_n < \beta_m$, *whenever* $|n| > |m| > 1$.

Proof of Step **??**. Without loss of generality, we assume that $n > 1$. We illustrate α_n and β_n on Figure **??**.

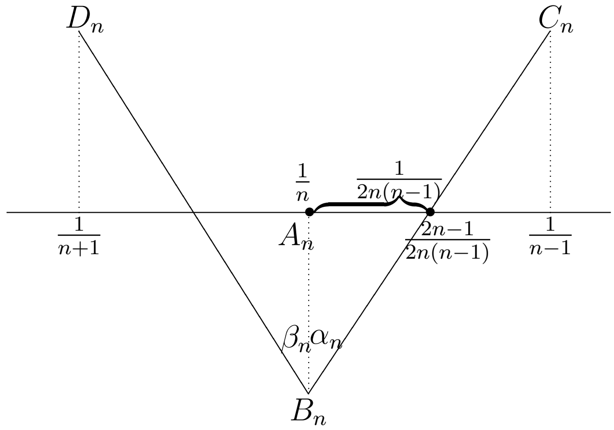

FIGURE **2.2.**

It is not difficult to compute that, as indicated on Figure **??**,

$$\sin\alpha_n = \frac{\frac{1}{2n(n-1)}}{\sqrt{1 + (\frac{1}{2n(n-1)})^2}} = \frac{1}{\sqrt{4n^2(n-1)^2 + 1}}.$$

Thus,

$$\frac{1}{2n(n-1)} = \frac{1}{\sqrt{4n^2(n-1)^2}} \geq \sin\alpha_n \geq \frac{1}{\sqrt{9n^2(n-1)^2}} = \frac{1}{3n(n-1)}.$$

Similarly,

$$\sin\beta_n = \frac{\frac{1}{2n(n+1)}}{\sqrt{1 + (\frac{1}{2n(n+1)})^2}} = \frac{1}{\sqrt{4n^2(n+1)^2 + 1}}.$$

Thus

$$\frac{1}{2n(n+1)} = \frac{1}{\sqrt{4n^2(n+1)^2}} \geq \sin\beta_n \geq \frac{1}{\sqrt{9n^2(n+1)^2}} = \frac{1}{3n(n+1)}.$$

The moreover part is clear from the expressions for $\sin\alpha_n$ and $\sin\beta_n$. □

Step 2. *If* $x = (x_1, x_2) \in K$ *and* $|x_1| \geq 1$ *then for* $r \leq \frac{1}{10}$

$$f(B(x, r)) \supset B(f(x), \frac{r}{6}).$$

Proof of Step **??**. Our assumptions imply that $x \in I_+ \cup I_-$, say $x \in I_+$, the case $x \in I_-$ follows by an identical argument. Then $x_t = (x_1, -1+t) \in B(x,r)$ for all $t \in (-r, r)$, and when $t < 0$, $f(x_t) = t$. Thus $f(B(x,r)) \supset (-r, 0]$. When $t > 0$, then, as illustrated on Figure **??**, by Step **??**,

$$f(x_t) = d(x_t, K) \geq d((1, -1+t), K) = t \sin \alpha_2 \geq t \cdot \frac{1}{6}.$$

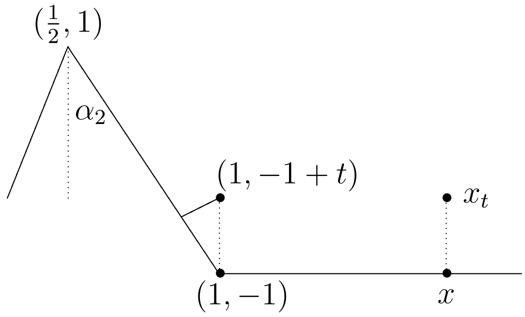

FIGURE **2.3.**

Thus $f(B(x,r)) \supset (-r, \frac{r}{6})$, which ends the proof. □

Step 3. *If* $x = (x_1, x_2) \in K$ *and* $x_1 = 0$ *then for* $r \leq \frac{1}{10}$

$$f(B(x,r)) \supset B(f(x), \frac{r^3}{2000}). \tag{2.1}$$

Proof of Step **??**. Since $x \in I_0$ we obtain that $|x_2| \leq 1$, as illustrated on Figure **??**.

Let n be the smallest odd number so that

$$\frac{1}{n-1} \leq \frac{r}{2}. \tag{2.2}$$

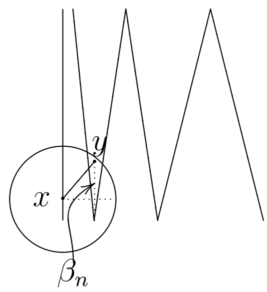

FIGURE **2.4.**

Then there exists $y = (y_1, y_2) \in B(x, r)$ so that $y_1 = \frac{1}{n}$ and $y_2 \geq x_2 + \frac{r}{2}$. Then

$$f(y) = d(y, K) \geq \frac{r}{2} \sin \beta_n \geq \frac{r}{2} \cdot \frac{1}{3n(n+1)}.$$

By (**??**) and since $r \leq \frac{1}{10}$ we see that $n > 4$ and $\frac{1}{n-3} > \frac{r}{2}$. Thus

$$\frac{1}{n} = \frac{n-3}{n} \cdot \frac{1}{n-3} \geq \frac{1}{4} \cdot \frac{r}{2},$$
$$\frac{1}{n+1} = \frac{n-3}{n+1} \cdot \frac{1}{n-3} \geq \frac{1}{5} \cdot \frac{r}{2}.$$

Hence

$$f(y) = d(y, K) \geq \frac{r}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{r}{2} \cdot \frac{1}{5} \cdot \frac{r}{2} = \frac{r^3}{480}.$$

Similarly there exists $z = (z_1, z_2) \in B(x, r)$ so that $z_1 = \frac{1}{n+1}$ and $z_2 \leq x_2 - \frac{r}{2}$. Then $f(z) = -d(z, K)$ and

$$d(z, K) \geq \frac{r}{2} \sin \beta_{n+1} \geq \frac{r}{2} \cdot \frac{1}{3(n+1)(n+2)} \geq \frac{r}{2} \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{r}{2} \cdot \frac{1}{6} \cdot \frac{r}{2} = \frac{r^3}{720}.$$

Thus (**??**) is satisfied. □

Step 4. *If $x = (x_1, x_2) \in K$ and $0 < |x_1| \leq 1$ then for $r \leq \frac{1}{10}$,* (**??**) *is satisfied.*

Proof of Step **??**. Since $x_1 \neq 0$ and $|x_1| \leq 1$, thus there exists $m \in \mathbb{Z}$ so that x belongs to I_m, the segment which connects points $(\frac{1}{m}, (-1)^m)$ and $(\frac{1}{m+1}, (-1)^{m+1})$. Without loss of generality we will assume that $m > 0$. Let n denote the smallest odd number so that

$$\frac{1}{n-1} \leq \frac{r}{2}.$$

If $m \geq n+1$ we proceed in a way very similar to Step **??**. Since $x_1 \in [\frac{1}{m+1}, \frac{1}{m}]$, we see that $0 < x_1 \leq \frac{1}{n+1}$, and thus there exist $y = (y_1, y_2) \in B(x, r)$ and $z = (z_1, z_2) \in B(x, r)$ so that $y_1 = \frac{1}{n}$, $y_2 \geq x_2 + \frac{r}{2}$, $z_1 = \frac{1}{n+1}$, $z_2 \leq x_2 - \frac{r}{2}$. Then $f(y) = d(y, K)$ and $f(z) = -d(z, K)$. Further, similarly as in Step **??**,

$$d(y, K) \geq \frac{r^3}{480},$$
$$d(z, K) \geq \frac{r^3}{720}.$$

Thus (**??**) is satisfied.

If $n > m - 1$ and $m > 3$ (the case when $m \leq 3$ is done similarly and we leave the details to the interested reader) then

$$\frac{1}{m-3} > \frac{r}{2}. \tag{2.3}$$

Now set $t = d(x, (\frac{1}{m}, (-1)^m))$ and let $y = (y_1, y_2)$ be the point with $y_1 = \frac{1}{m}$, so that the segment $[x, y]$ with endpoints x and y is perpendicular to I_m, see Figure **??**.

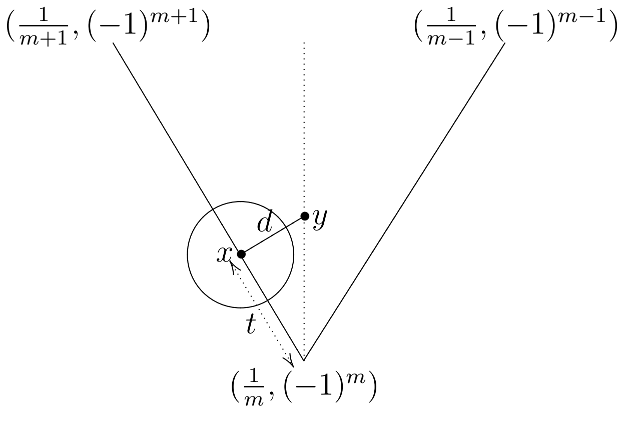

FIGURE **2.5.**

If $t \geq \frac{r}{3}$ then, by (**??**), (since $m > 3$),

$$d = d(x,y) = t \cdot \sin\beta_m \geq \frac{r}{3} \cdot \frac{1}{3m(m+1)} \geq \frac{r}{3} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{r}{2} \cdot \frac{1}{5} \cdot \frac{r}{2} = \frac{r^3}{720}.$$

If $d < r$ then $y \in B(x,r)$ and $f(y) = d \geq \frac{r^3}{720}$. If $d \geq r$, let y_r denote a point in the segment with endpoints x and y so that $d(x, y_r) = r$. Then $f(y_r) = r$ and $f(B(x,r)) \supset f([x, y_r)) \supset [0, r)$.

Next we consider the case when $t < \frac{r}{3}$, as illustrated on Figure **??**

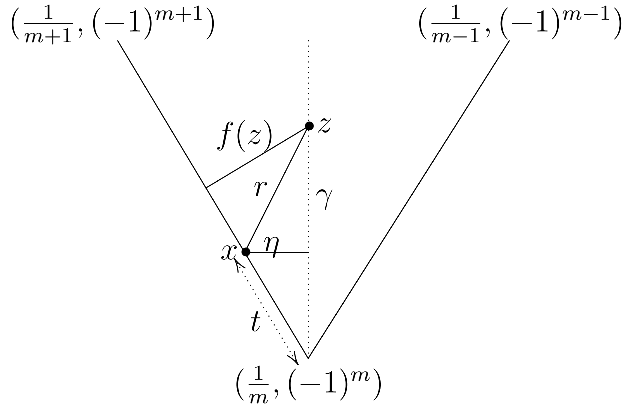

FIGURE **2.6.**

Then

$$\eta = d(x, \{v = (v_1, v_2) : v_1 = \frac{1}{m}\}) = t \sin\beta_m \leq \frac{r}{3} \cdot \frac{1}{m(m+1)} \leq \frac{r}{3}.$$

Hence

$$\gamma = \sqrt{r^2 - \eta^2} \geq \sqrt{r^2 - \frac{r^2}{9}} = r\frac{\sqrt{8}}{3}.$$

Thus there exists $z \in B(x,r)$ with $z_1 = \frac{1}{m}$ and $z_2 \geq x_2 + r\frac{\sqrt{8}}{3} \geq -1 + \frac{2}{3}r$. We have

$$f(z) = d(z, K) \geq \frac{2}{3} \cdot r \cdot \sin\beta_m \geq \frac{2}{3} r \cdot \frac{1}{3m(m+1)} \geq \frac{2}{3} \cdot r \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{r}{2} \cdot \frac{1}{5} \cdot \frac{r}{2} = \frac{r^3}{360}.$$

Thus for all $t \in [0,1]$ we conclude that $f(B(x,r)) \supset [0, \frac{r^3}{720}]$.

A similar computation shows that $f(B(x,r))$ contains also a sufficiently large negative interval, so that (**??**) holds. □

Step 5. *If $x = (x_1, x_2)$ is such that $|x_1| \geq 1$ and $d(x, K) = d > 0$, then for $r \leq \frac{1}{10}$*

$$f(B(x,r)) \supset B(f(x), \frac{r}{24}).$$

Proof of Step **??**. We will assume without loss of generality that $f(x) > 0$ and $x_1 > 1$, the other cases follow by analogous arguments.

If $r \leq d$, then $B(x,r) \cap K = \emptyset$ and, since $|x_1| \geq 1$, this implies that $f(B(x,r)) \supset (d - r, d + r)$. Hence also if $\frac{r}{4} \leq d$ then

$$f(B(x,r)) \supset f(B(x, \frac{r}{4})) \supset (d - \frac{r}{4}, d + \frac{r}{4}).$$

Next we assume that $r \geq 4d$. Since $r \leq \frac{1}{10}$ and $f(x) > 0$, we conclude that $x_2 \in (-1, 1)$. Let $x' = (1, x_2)$ and $d' = d(x', K)$, cf. Figure **??**.

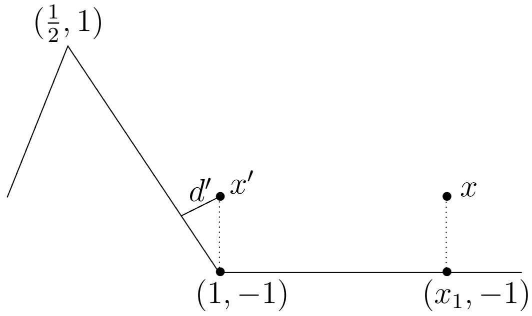

FIGURE **2.7.**

Then $d' \leq d$ and

$$\frac{d'}{x_2+1} = \frac{\frac{1}{2}}{\sqrt{2^2+(\frac{1}{2})^2}},$$

$$d' = \frac{1}{\sqrt{5}}(x_2+1).$$

Thus

$$d((x_1,-1),x) = x_2+1 \leq \sqrt{5}d' \leq \sqrt{5}d \leq \frac{\sqrt{5}}{4}r \leq \frac{3}{4}r,$$

and therefore, by Step **??**,

$$f(B(x,r)) \supset f(B((x_1,-1),\frac{r}{4})) \supset (-\frac{r}{24},\frac{r}{24}).$$

Since we also have that $f(B(x,r)) \supset (0,d+r)$, hence we obtain

$$f(B(x,r)) \supset (-\frac{r}{24},d+r) \supset (d-\frac{r}{24},d+\frac{r}{24}),$$

as claimed. □

Step 6. *If $x=(x_1,x_2)$ is such that $|x_1| \leq 1$ and $d(x,K)=d>0$ then for all $r \leq \frac{1}{10}$:*

(a) *if $f(x)>0$ then*

$$f(B(x,r)) \supset (\max(d-r,0),d+\frac{r^3}{480}); \tag{2.4}$$

(b) *if $f(x)<0$ then*

$$f(B(x,r)) \supset (-d-\frac{r^3}{480},\min(r-d,0)). \tag{2.5}$$

Proof of Step **??**. We will assume without loss of generality that $f(x)>0$. The case when $f(x)<0$ is proven identically.

Let $z=(z_1,z_2) \in K$ denote a point in K such that $d(x,z)=d(x,K)=d$. Note that, if $r>d$ then $z \in B(x,r)$ and thus $[0,d] \subset f(B(x,r))$. If $r \leq d$ then $B(x,r)$ contains a subinterval of length r of the interval $[x,z]$ and $f(B(x,r)) \supset (d-r,d]$. Thus to prove (**??**), it is enough to prove that $f(B(x,r)) \supset [d,d+\frac{r^3}{480})$.

For this we will consider two cases.

CASE 1. First we assume that $z_2 \leq 1$. This, together with the fact that $x_1 \neq 0$, implies that $z_1 \neq 0$. Moreover, since $f(x)>0$, there exists the unique odd $m \in \mathbb{Z}$ (say, $m>0$) so that $x_1 \in (\frac{1}{m+1},\frac{1}{m-1})$ (this means that x lies "above" the set $I_m \cup I_{m-1}$), see Figure **??**.

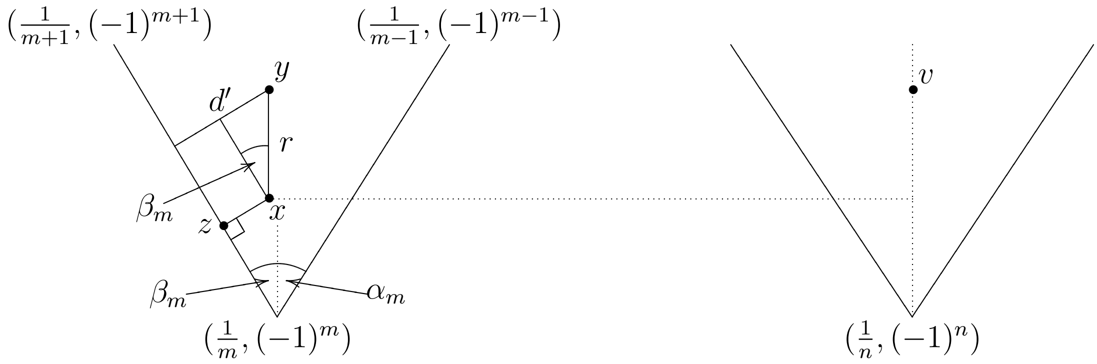

FIGURE **2.8.**

If $\frac{1}{m+1} \geq \frac{r}{4}$ we consider a point $y = (x_1, x_2 + r) \in \overline{B(x,r)}$. Let d' denote the distance between y and the straight line containing I_m if $z \in I_m$, or the straight line containing I_{m-1} if $z \in I_{m-1}$. Then

$$\begin{aligned} f(y) = d(y,K) \geq d' &\geq d + r \sin \beta_m \\ &\geq d + r\frac{1}{3m(m+1)} \geq d + r \cdot \frac{1}{3} \cdot \frac{r}{4} \cdot \frac{r}{4} \\ &= d + \frac{r^3}{48}. \end{aligned}$$

If $\frac{1}{m+1} < \frac{r}{4}$ then we proceed similarly to Steps **??** and **??**. Let n be the smallest number so that $(n-m)$ is even and

$$\frac{1}{n-1} \leq \frac{r}{2}.$$

Note that since $r \leq \frac{1}{10}$, we have $r \geq 40$. Further

$$\begin{gathered} \frac{1}{n-3} > \frac{r}{2} = 2\frac{r}{4} > \frac{2}{m+1}, \\ 2(n-3) < m+1, \\ n < \frac{m+7}{2} < m, \end{gathered}$$

since $m > 7$. Hence $\frac{1}{m-1} \leq \frac{1}{n}$ and since $x_1 \in (\frac{1}{m+1}, \frac{1}{m-1})$ we get $0 < x_1 < \frac{1}{n}$, and therefore that there exists $v = (v_1, v_2) \in B(x,r)$ so that $v_1 = \frac{1}{n}, v_2 \geq x_2 + \frac{r}{2}$. Note that

$$d = d(x,K) \leq (x_2+1)\sin\beta_m < (x_2+1)\sin\beta_n,$$

since $n < m$ (cf. Step **??**). Further, we have:

$$\begin{aligned} f(v) = d(v,K) &\geq d(v, (\frac{1}{n}, (-1)^n)) \cdot \sin\beta_n \\ &\geq (x_2 + \frac{r}{2} + 1)\sin\beta_n \geq (x_2+1)\sin\beta_m + \frac{r}{2}\sin\beta_n \\ &\geq d + \frac{r}{2} \cdot \frac{1}{3n(n+1)} \geq d + \frac{r^3}{480}. \end{aligned}$$

Thus (**??**) is satisfied.

CASE 2. Next we consider the case when $z_2 = 1$. This implies that $x_2 > 1$. Further, if $y = (x_1, x_2 + r) \in \overline{B(x,r)}$, then $d(y, K) = d(y, z)$ and $|\sphericalangle yxz| \geq \frac{\pi}{2}$ (cf. Figure **??**). Thus

$$f(y) = d(y, z) \geq \sqrt{d^2 + r^2}. \tag{2.6}$$

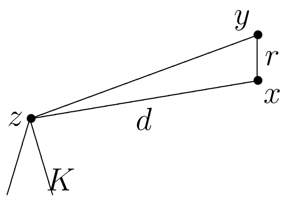

FIGURE **2.9.**

Therefore, if $\frac{1}{10} \leq r \leq d \leq 10 \leq \frac{1}{r}$, we obtain

$$\begin{aligned} f(y) &\geq \sqrt{d^2 + r^2} = d\sqrt{1 + \frac{r^2}{d^2}} \geq d(1 + \frac{1}{2} \cdot \frac{r^2}{d^2}) \\ &= d + \frac{1}{2} \cdot \frac{r^2}{d} \geq d + \frac{1}{2} \cdot \frac{r^2}{r} = d + \frac{r}{2}, \end{aligned}$$

so (**??**) holds.

If $d \leq r$, then (**??**) implies that

$$\begin{aligned} f(y) &\geq \sqrt{d^2 + r^2} = r\sqrt{1 + \frac{d^2}{r^2}} \geq r(1 + \frac{1}{2} \cdot \frac{d^2}{r^2}) \\ &= r + \frac{1}{2} \cdot \frac{d^2}{r}. \end{aligned} \tag{2.7}$$

Therefore, if $r \geq d \geq \frac{r^2}{\sqrt{240}}$, we obtain

$$f(y) \geq r + \frac{1}{2} \cdot \frac{r^4}{240r} \geq d + \frac{r^3}{480}.$$

On the other hand, when $d \leq r - \frac{r^3}{480}$, then (**??**) implies

$$f(y) \geq r + \frac{1}{2} \cdot \frac{d^2}{r} \geq r \geq d + \frac{r^3}{480}.$$

We now note that for all $r \leq \frac{1}{10}$,

$$r - \frac{r^3}{480} \geq \frac{r^2}{\sqrt{240}}.$$

Indeed, for $r \in (0, \frac{1}{10}]$,

$$\frac{r^2}{\sqrt{240}} + \frac{r^3}{480} - r = r(\frac{r}{\sqrt{240}} + \frac{r^2}{480} - 1) < 0,$$

since for $r \in (0, \frac{1}{10}]$,

$$-1 < \frac{r}{\sqrt{240}} + \frac{r^2}{480} - 1 < 0.$$

Therefore (**??**) is satisfied for all $d \leq r$.

To finish the proof of CASE 2, we now assume that $d \geq 10$. Since $|x_1| \leq 1$, this implies that $x_2 \geq 8$. Thus $|\sphericalangle xyz| < \frac{\pi}{4}$, and the position of points x, y and set K looks more like

Figure **??** below (please note that the segment $[z, x]$ is not drawn to scale, d is much longer than it appears).

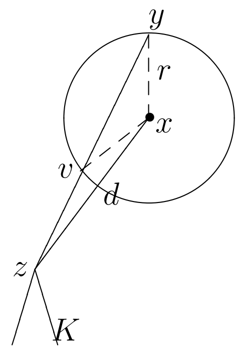

FIGURE **2.10.**

Let $v \in \overline{B(x,r)}$ be so that $d(v,x) = r$ and v lies on the segment $[y, z]$. Since $|\sphericalangle xyz| < \frac{\pi}{4}$, we conclude that $|\sphericalangle vxy| > \frac{\pi}{2}$ and therefore $d(v,y) > \sqrt{2}r$. Hence we obtain

$$f(y) = d(y,z) = d(v,z) + d(v,y) \geq (d-r) + \sqrt{2}r \geq d + \frac{r}{3}.$$

Thus (**??**) holds when $d \geq 10$, which ends the proof of CASE 2 and of this step. ☐

As an immediate corollary of Step **??** we obtain the following:

Step 7. *If* $d(x,K) = d > 0$ *and* $r \leq \min(\frac{1}{10}, d)$ *then*

$$f(B(x,r)) \supset B(f(x), \frac{r^3}{480}).$$

Step 8. *If* $d(x,K) = d > 0$ *and* $\frac{1}{10} \geq r > d$ *then*

$$f(B(x,r)) \supset B(f(x), \frac{r^3}{16000}). \tag{2.8}$$

Proof of Step **??**. We start from the trivial observation that when $\frac{1}{10} \geq r > d$ then, by Step **??**,

$$f(B(x,r)) \supset f(B(x,d)) \supset B(f(x), \frac{d^3}{480}).$$

Thus, if

$$\frac{r^3}{16000} \leq \frac{d^3}{480}, \tag{2.9}$$

then (**??**) is satisfied. Equation (**??**) is true when $r \leq 2d$. Thus, next we assume that

$$\frac{1}{10} \geq r \geq 2d.$$

We will also assume, without loss of generality, that $f(x) > 0$. Let $y \in K$ be such that $d(x,y) = d$. Then $B(x,r) \supset B(y, r-d)$. By Steps **??** and **??**, (**??**) holds and we have

$$f(B(y, r-d)) \supset (-\frac{(r-d)^3}{2000}, 0].$$

Note that

$$\frac{(r-d)^3}{2000} \geq \frac{(\frac{1}{2}r)^3}{2000} = \frac{r^3}{16000}.$$

Thus

$$f(B(x,r)) \supset (-\frac{r^3}{16000}, d].$$

On the other hand, by Step **??**,

$$f(B(x,r)) \supset (0, d + \frac{r^3}{480}).$$

Thus (**??**) is satisfied. □

This ends the proof that f is co-uniformly continuous with the modulus ω. □

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