ON THE STRUCTURE OF LEVEL SETS OF UNIFORM AND LIPSCHITZ QUOTIENT MAPPINGS FROM $\mathbb{R}^n$ TO $\mathbb{R}$

BEATA RANDRIANANTOANINA*

Abstract. We study two questions posed by Johnson, Lindenstrauss, Preiss, and Schechtman, concerning the structure of level sets of uniform and Lipschitz quotient mappings from $\mathbb{R}^n \to \mathbb{R}$. We show that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n \geq 2$, is a uniform quotient mapping then for every $t \in \mathbb{R}$, $f^{-1}(t)$ has a bounded number of components, each component of $f^{-1}(t)$ separates $\mathbb{R}^n$ and the upper bound of the number of components depends only on n and the moduli of co-uniform and uniform continuity of f.

Next we prove that all level sets of any co-Lipschitz uniformly continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}$ are locally connected, and we show that for every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r\to 0} \Omega(r) = 0$, there exists a natural number $M = M(c, \Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω, there exists a natural number $n(f) \leq M$ and a finite set $T_f \subset \mathbb{R}$ with $\text{card}(T_f) \leq n(f) - 1$ so that for all $t \in \mathbb{R} \setminus T_f$, $f^{-1}(t)$ has exactly $n(f)$ components, $\mathbb{R}^2 \setminus f^{-1}(t)$ has exactly $n(f) + 1$ components and each component of $f^{-1}(t)$ is homeomorphic with the real line and separates the plane into exactly 2 components. The number and form of components of $f^{-1}(s)$ for $s \in T_f$ are also described – they have a finite tree structure.

1. Introduction

Let X, Y be metric spaces. A mapping $f : X \longrightarrow Y$ is said to be a *uniform quotient mapping* if there exist functions $\omega, \Omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(r) > 0$ for all $r > 0$ and $\lim_{r\to 0} \Omega(r) = 0$ so that for all $x \in X$ and all $r > 0$:

$$B(f(x), \omega(r)) \subset f(B(x, r)) \subset B(f(x), \Omega(r)), \tag{1.1}$$

where $B(x, r)$ denotes the open ball with center x and radius r.

Notice that the right hand inclusion means that f is uniformly continuous. The mapping f is called *co-uniformly continuous* if the left hand inclusion in (1.1) is satisfied. There is no restriction in assuming that the functions ω and Ω are continuous and increasing. They are called *moduli of co-uniform and uniform continuity of* f, respectively. If the functions ω and Ω are linear, i.e. if there exist constants $c, L > 0$ so that for all $x \in X$ and all $r > 0$:

$$B(f(x), cr) \subset f(B(x, r)) \subset B(f(x), Lr), \tag{1.2}$$

then f is called a *Lipschitz quotient mapping*. Clearly the right hand inclusion in (1.2) means that f is a Lipschitz mapping. If f satisfies the left hand inclusion of (1.2), f is called a *co-Lipschitz mapping*. Constants c and L are called *co-Lipschitz and Lipschitz constants of* f, respectively. The study of uniform and Lipschitz quotient mappings was initiated in [1], see also [3] for the comprehensive introduction of the subject. The structure of Lipschitz and uniform quotient mappings $f : X \longrightarrow Y$, when X and Y are finite dimensional was studied by Johnson, Lindenstrauss, Preiss and Schechtman in [7]. They obtained most complete

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*Participant, NSF Workshop in Linear Analysis and Probability, Texas A&M University.

results for the case of $X = Y = \mathbb{R}^2$. For $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ they proved, in particular, that if f is uniformly continuous and co-Lipschitz, e.g. if f is a Lipschitz quotient mapping, then for every $t \in \mathbb{R}^2$, $f^{-1}(t)$ is a finite set of points in $\mathbb{R}^2$ and $f = P \circ h$ where h is a homeomorphism of the plane and P is a complex polynomial (see also Remark 5.2 below). The question whether level sets of $f^{-1}(t)$ of a Lipschitz quotient map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are discrete, is open for all $n > 2$.

In [7], the authors also study the structure of level sets $f^{-1}(t)$ of uniform and Lipschitz maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. They showed, among others, the following results:

Theorem 1.1. [7, Proposition 5.1] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a uniform quotient mapping satisfying (1.1). Then for each $t \in \mathbb{R}$ the number of components of $\mathbb{R}^n \setminus f^{-1}(t)$ is finite and bounded by a function of n, $\omega(\cdot)$ and $\Omega(\cdot)$ only.*

Theorem 1.2. [7, Proposition 5.4] *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a Lipschitz quotient mapping. Then, for each $t \in \mathbb{R}$, each component of $f^{-1}(t)$ is unbounded and separates the plane.*

Theorem 1.3. [7, Corollary 5.5] *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a Lipschitz quotient mapping. Then, for each $t \in \mathbb{R}$, $f^{-1}(t)$ has a bounded number of components. The upper bound of the number of components depends only on the Lipschitz and co-Lipschitz constants of f.*

They also asked the following two questions:

(Q1) Can one weaken the assumption of Lipschitz quotient to uniform quotient mappings in Theorems 1.2 and 1.3?

(Q2) To what extent is the number of components of $f^{-1}(t)$ or of $\mathbb{R}^2 \setminus f^{-1}(t)$ independent of t? Are these numbers constant after excluding finitely many values of t?

Question (Q2) is motivated by the following two examples of Lipschitz quotient mappings from $\mathbb{R}^2$ to $\mathbb{R}$. In both cases the mapping f is the ℓ_1 distance from the solid lines multiplied, in each component of the complement of the solid lines, by the sign indicated.

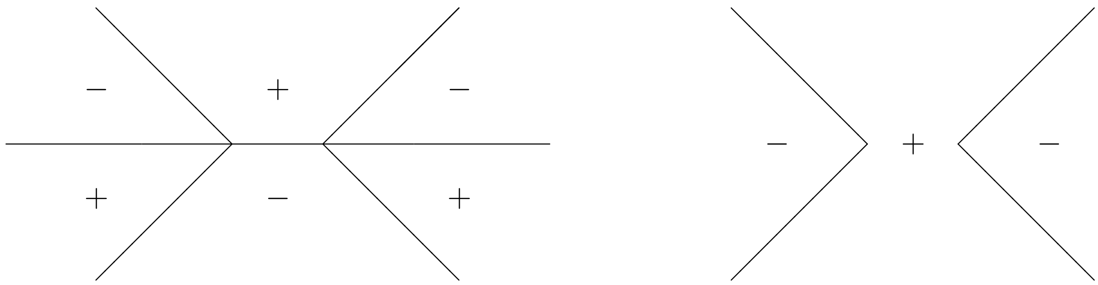

FIGURE **1.1.**

Here $f^{-1}(0)$ has one component in the first example and two in the second, and $\mathbb{R}^2 \setminus f^{-1}(0)$ has six and three components, respectively. The authors of [7] note that it is easy to draw examples with an arbitrary finite number of components of $f^{-1}(0)$. Thus question (Q2) is essentially asking whether all Lipschitz quotient maps $\mathbb{R}^2$ to $\mathbb{R}$ have the form similar to the examples illustrated in Figure 1.1.

This paper is devoted to the study of questions (Q1) and (Q2). We answer both of them affirmatively. First, in Section 2, we obtain generalizations of Theorems 1.3 and 1.2 for uniform quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}$ for any $n \geq 2$ (Theorem 2.4) and Corollary 2.5, respectively). Our results follow from Theorem 1.1 through general topological arguments based on the Phragmen-Brower theorem and the theory of separation in $\mathbb{R}^n$.

Next we study question (Q2). We obtain not only information about the number of components of $f^{-1}(t)$ and of $\mathbb{R}^2 \setminus f^{-1}(t)$ for Lipschitz quotient maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, but we give the full characterization of both the number and the form of each component of any level set $f^{-1}(t)$ of co-Lipschitz uniformly continuous mappings $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (Theorem 5.1). We show that for every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r\to 0} \Omega(r) = 0$, there exists a natural number $M = M(c, \Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω, there exists a natural number $n(f) \le M$ and a finite set $T_f \subset \mathbb{R}$ with $\mathrm{card}(T_f) \le n(f) - 1$ so that for all $t \in \mathbb{R} \setminus T_f$, $f^{-1}(t)$ has exactly $n(f)$ components, $\mathbb{R}^2 \setminus f^{-1}(t)$ has exactly $n(f) + 1$ components and each component of $f^{-1}(t)$ is homeomorphic with the real line and separates the plane into exactly 2 components. The number and form of components of $f^{-1}(s)$ for $s \in T_f$ is also described – these components have a finite tree structure (for precise formulation see Theorems 5.1, 4.11 and Remark 5.4).

Thus we do confirm that co-Lipschitz uniformly continuous mappings from $\mathbb{R}^2$ to $\mathbb{R}$ have a form analogous to the examples presented on Figure 1.1. Moreover, we prove that, as on Figure 1.1, no level set $f^{-1}(t)$ can contain parallel lines, but the distance between unbounded components of $f^{-1}(t) \setminus B(0, R)$ has to increase to infinity as R increases to infinity, cf. Figure 1.2 (Corollary 5.12, cf. also Remark 5.5).

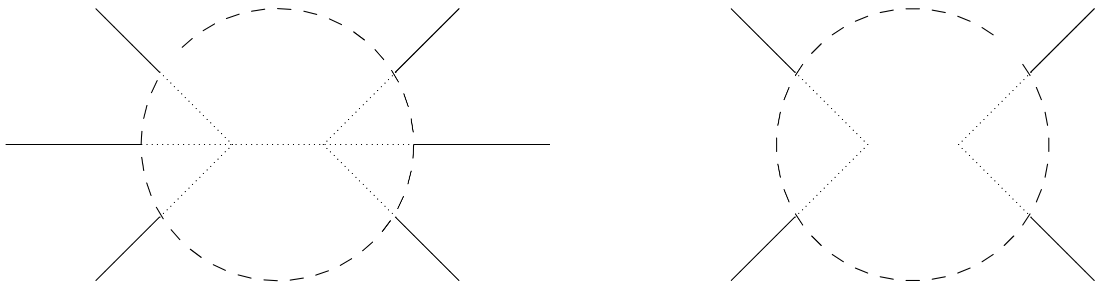

FIGURE **1.2.**

Our method of proof of Theorem 5.1 depends on a careful analysis of topological properties of level sets $f^{-1}(t)$, their end points and their structure at infinity. The crucial property that we use in a very essential way is the fact that level sets $f^{-1}(t)$ are locally connected when f is a co-Lipschitz uniformly continuous map from $\mathbb{R}^2$ to $\mathbb{R}$ (Proposition 3.5).

We do not know whether level sets of co-Lipschitz uniformly continuous maps or of Lipschitz quotient maps from $\mathbb{R}^n$ to $\mathbb{R}$ are locally connected when $n > 2$. If one looks for a counter-example, the most natural map to check would be the Lipschitz quotient map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ constructed by Csörnyei [4], whose level set $f^{-1}(0)$ is very large and complicated. It turns out, however, that for this map and also for its both coordinate maps, which go from $\mathbb{R}^3$ to $\mathbb{R}$, all level sets are locally connected.

However we do know that there exist uniform quotient maps from $\mathbb{R}^2$ to $\mathbb{R}$ with non-locally connected level sets (see Example 3.6).

The local connectedness of level sets $f^{-1}(t)$ of a co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, allows us to use the notion of ends from the algebraic topology (cf. [6], see Definition 4.10) to analyze the behavior of level sets at infinity and consequently to fully describe the topological structure of level sets and their complements (which is achieved in Sections 4 and 5).

Throughout the paper we use standard notation, as may be found in [3, 8, 9, 20].

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2. Number of components of level sets of uniform quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}$ is finite

As a corollary of Theorem 1.1 using purely topological arguments we will show that when $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a uniform quotient mapping then for each $t \in \mathbb{R}$, the number of components of $f^{-1}(t)$ is finite (Theorem 2.4 below). Our main tool is the following fact:

Theorem 2.1. *Let $B_0, B_1 \subset \mathcal{S}_n$, $n \geq 2$, be two closed sets such that $B_0 \cap B_1 \subseteq \{q\}$ a singlepoint. If none of the sets B_0 or B_1 separates between points p_1 and p_2 then their union $B_0 \cup B_1$ does it neither.*

The above statement combines [9, Theorem 59.II.11 and 61.I.7] specialized to the situation in the present paper. In the case when $n = 2$, Kuratowski refers to this fact as the *first theorem of Janiszewski*, and it's general version is called the *Phragmen-Brouwer theorem.*

Although the subject is closely related to some classical duality theorems, cf. [14, 17, 2], we were unable to find in the literature results that we could directly use in the situation we deal with. We decided to present a proof of the fact we needed, based on some standard arguments concerning separation in $\mathbb{R}^n$.

We start from two lemmas.

Lemma 2.2. *Let A be an open connected subset of $\mathcal{S}_n$, so that $\overline{A} \neq \mathcal{S}_n$ and $\mathrm{Bd}(A) = F_1 \cup F_2$ where F_1, F_2 are closed sets with $F_1 \cap F_2 \subseteq \{q\}$ a singlepoint. Let $p_1 \in A$ and $p_2 \notin \overline{A}$. Then exactly one of the sets F_1 or F_2 separates between p_1 and p_2.*

Proof. By Theorem 2.1 we conclude that at least one of the sets F_1 or F_2 separates p_1 and p_2. Suppose now that each of F_1 and F_2 separates between p_1 and p_2. Then there exist components C_1, C_2 of $\mathcal{S}_n \setminus F_1$, $\mathcal{S}_n \setminus F_2$ respectively so that

$$p_1 \in C_1 \cap C_2, \text{ and thus } A \subset C_1 \cap C_2,$$
$$p_2 \notin C_1 \cup C_2.$$

Then

$$\mathrm{Bd}(C_1 \cup C_2) \subset \mathrm{Bd}(C_1) \cup \mathrm{Bd}(C_2) \subset F_1 \cup F_2.$$

Let $x \in F_1 \setminus \{q\}$. Then for every neighborhood V_x of x we have $V_x \cap A \neq \emptyset$, since $x \in \mathrm{Bd}(A)$. Thus $V_x \cap C_2 \neq \emptyset$ and $x \in \overline{C}_2$. Since $x \notin F_2$ we conclude that $x \in C_2$ and therefore $x \notin \mathrm{Bd}(C_1 \cup C_2)$. Similarly, if $y \in F_2 \setminus \{q\}$ then $y \notin \mathrm{Bd}(C_1 \cup C_2)$. Thus $\mathrm{Bd}(C_1 \cup C_2) \subset \{q\}$ which contradicts the fact that $p_2 \notin C_1 \cup C_2$. □

Lemma 2.3. *Let A be an open connected subset of $\mathcal{S}_n$ so that $A \neq \mathcal{S}_n$ and $\mathrm{Bd}(A) = F_1 \cup F_2$ where F_1, F_2 are closed sets with $F_1 \cap F_2 \subseteq \{q\}$ a singlepoint. Suppose that $\mathcal{S}_n \setminus F_1$ is connected. Then for every $x \in F_1 \setminus \{q\}$ there exists a neighborhood U_x of x so that $U_x \subset \overline{A}$.*

Proof. Let $x \in F_1 \setminus \{q\}$. Since F_2 is closed, there exists a connected neighborhood U_x of x so that $U_x \cap F_2 = \emptyset$. Since $x \in \mathrm{Bd}(A)$, there exists $y \in U_x$ so that $y \in A$. Suppose that

$U_x \setminus \overline{A} \neq \emptyset$ and let $z \in U_x \setminus \overline{A}$. Then $\mathrm{Bd}(A)$ separates between the points y and z. But $\mathcal{S}_n \setminus F_1$ is connected so F_1 does not separate between y and z. Thus by Theorem 2.1 we conclude that F_2 separates between y and z. But this is a contradiction since y, z belong to a connected set U_x which is disjoint with F_2. □

With these tools we are ready to prove the main theorem of this section.

Theorem 2.4. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a uniform quotient map. Then, for any $t \in \mathbb{R}$, a number of connected components of $f^{-1}(t)$ is finite and bounded by a function of n, $\omega(\cdot)$ and $\Omega(\cdot)$ only.*

Proof. We consider $\mathbb{R}^n$ as embedded in its one point compactification $\mathcal{S}_n$. Denote $K = f^{-1}(t)$. By [7, Lemma 5.2], K is unbounded and therefore the closure of K in $\mathcal{S}_n$ equals $K \cup \{\infty\}$, and the closure in $\mathcal{S}_n$ of every component of K contains $\{\infty\}$. By Theorem 1.1, $\mathbb{R}^n \setminus K$ and therefore also $\mathcal{S}_n \setminus \overline{K}$ has a finite number of components, say

$$\mathcal{S}_n \setminus \overline{K} = \bigcup_{j=1}^{m} C_j.$$

Here $C_j \subset \mathcal{S}_n \setminus \{\infty\}$, so each C_j can also be considered as a subset of $\mathbb{R}^n$. Note that C_j cannot be bounded in $\mathbb{R}^n$, so $\infty \in \mathrm{Bd}(C_j) \subset \mathcal{S}_n$ for all j. Suppose that there exists j, say $j = 1$, so that $\mathrm{Bd}(C_1)$ has m or more connected components in $\mathbb{R}^n$. Then $\mathrm{Bd}(C_1) \subset \mathbb{R}^n$ can be presented as a sum of m disjoint closed sets in $\mathbb{R}^n$, which are not necessarily connected. Thus after taking closures in $\mathcal{S}_n$ we see that

$$\mathrm{Bd}(C_1) = F_1 \cup \cdots \cup F_m,$$

where $\{F_k\}_{k=1}^m$ are closed sets in $\mathcal{S}_n$, not necessarily connected, so that $F_k \cap F_l \subseteq \{\infty\}$ for all $k \neq l$.

Let $\{p_j\}_{j=1}^m$ be a collection of points such that $p_j \in C_j$ for $j = 1, \cdots, m$. Since for each $j = 2, \ldots, m$, $\mathrm{Bd}(C_1)$ separates between p_1 and p_j, by Theorem 2.1, there exists $\sigma(j) \in \{1, \cdots, m\}$ so that $F_{\sigma(j)}$ separates between p_1 and p_j. By Lemma 2.2, $\mathrm{Bd}(C_1) \setminus F_{\sigma(j)}$ does not separate between p_1 and p_j, so the choice of $\sigma(j)$ is unique. Thus $\mathrm{card}(\{\sigma(j)\}_{j=2}^m) \leq m - 1$. Hence there exists $j_0 \in \{1, \cdots, m\}$ so that F_{j_0} does not separate between p_1 and p_i for all $i = 2, \cdots, m$. Thus $\mathcal{S}_n \setminus F_{j_0}$ is connected, and by Lemma 2.3 for every $x \in F_{j_0} \setminus \{\infty\}$ there exists a neighborhood U_x of x so that $U_x \subset \overline{C}_1$. But $f(C_1) \subset (t, \infty)$ or $f(C_1) \subset (-\infty, t)$, thus $f(U_x) \subset [t, \infty)$ or $f(U_x) \subset (-\infty, t]$, which contradicts the fact that $f(U_x) \supset B(f(x), \varepsilon) = (t - \varepsilon, t + \varepsilon)$ for some $\varepsilon > 0$. This contradiction yields that $\mathrm{Bd}(C_1)$ has at most $(m-1)$ components in $\mathbb{R}^n$. Similarly, for every $j \in \{1, \ldots, m\}$, $\mathrm{Bd}(C_j)$ has at most $(m-1)$ components and since every component of K contains a component of $\mathrm{Bd}(C_j)$ for at least one $j \in \{1, \cdots, m\}$, we conclude that the number of components of K is smaller or equal than $m(m-1)$. □

Corollary 2.5. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a uniform quotient mapping. Then, for each $t \in \mathbb{R}$, each component of $f^{-1}(t)$ separates $\mathbb{R}^n$.*

This Corollary has word for word the same proof as [7, Proposition 5.4], since by Theorem 2.4, for each $t \in \mathbb{R}$, $f^{-1}(t)$ has a finite number of components.

3. Local connectedness of level sets

In this section we show that all level sets of co-Lipschitz uniformly continuous mappings from $\mathbb{R}^2$ to $\mathbb{R}$ are hereditarily locally connected. This is a very strong property which will enable us to give a detailed description of the structure of the level sets, see Sections 4 and 5. We begin by recalling some basic definitions.

Definition 3.1. A topological space S is said to be *locally connected at a point* x if for every open set U containing x there is a connected open set V so that $x \in V \subset U$. The space S is *locally connected* if it is locally connected at each point and S is *hereditarily locally connected* if every subcontinuum of S is locally connected.

We will use the following characterization of hereditary local connectedness:

Theorem 3.2. [20, V.(2.1) and I.(12.2)] *A locally compact connected set S is hereditarily locally connected if and only if S does not contain a continuum of convergence.*

Recall that if a continuum K is a subset of a set M then K is called a *continuum of convergence* of M provided that there exists in M a sequence of mutually exclusive continua $K_1, K_2, \ldots$, no one of which contains a point of K and which converges to K as a limit, i.e. $K \cap \bigcup_{i=1}^{\infty} K_i = \emptyset$ and $\lim[K_i]_i = K$.

Here $\lim[K_i]_i$ denotes the limit of a sequence $[K_i]_i$ which is defined as follows (cf. [20, Section I.7] or [8, Chapter 11, Section 29]): The set of all points x such that every neighborhood of x contains points of infinitely many sets of $[K_i]_i$ is called the *limit superior* of $[K_i]_i$ and is denoted $\limsup[K_i]_i$. The set of all points y such that every neighborhood of y contains points of all but a finite number of the sets $[K_i]_i$ is called the *limit inferior* of $[K_i]_i$ and is denoted $\liminf[K_i]_i$. If $\limsup[K_i]_i = \liminf[K_i]_i$ then we say that the collection $[K_i]_i$ is *convergent* and we write $\lim[K_i]_i = \limsup[K_i]_i = \liminf[K_i]_i$ and we call $\lim[K_i]_i$, the *limit* of $[K_i]_i$.

We will prove that if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a co-Lipschitz uniformly continuous mapping then for all $t \in \mathbb{R}$, $f^{-1}(t)$ does not contain a continuum of convergence. For this we will need the following "bottleneck lemma", whose proof is very similar to the proof of [7, Lemma 5.3].

Lemma 3.3. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map with co-Lipschitz constant 1 and a modulus of uniform continuity Ω. Let K_1, K_2 be disjoint subcontinua of $f^{-1}(0)$ and let $\alpha \in \mathbb{R}_+$. If there exist points $x_1, x_2 \in K_1$, $y_1, y_2 \in K_2$ so that, for $i = 1, 2$,*

$$d(x_i, y_i) \leq \alpha,$$

then

$$d(x_1, x_2) \leq 2\Omega(\frac{\alpha}{2}) + 4\alpha.$$

For the proof we will need the following basic lemma concerning the lifting of Lipschitz curves which was established in [1].

Lemma 3.4. [1, Lemma 4.4] *Suppose that $f : \mathbb{R}^n \longrightarrow X$ is continuous and co-Lipschitz with constant one, $f(x) = y$, and $\xi : [0, \infty) \longrightarrow X$ is a curve with Lipschitz constant one, and $\xi(0) = y$. Then there is a curve $\phi : [0, \infty) \longrightarrow \mathbb{R}^n$ with Lipschitz constant one such that $\phi(0) = x$ and $f(\phi(t)) = \xi(t)$ for $t \geq 0$.*

Proof of Lemma 3.3. If $d(x_1, x_2) \leq 2\alpha$ then we are done, so assume without loss of generality that $d(x_1, x_2) > 2\alpha$. For $i = 1, 2$, let I_i be the segment connecting x_i and y_i, i.e. $I_i =$

$\{(1-t)x_i + ty_i \ : \ t \in [0,1]\}$. Then $length(I_i) \leq \alpha$, for $i = 1, 2$, and thus, if $d(x_1, x_2) > 2\alpha$ then $I_1 \cap I_2 = \emptyset$. Set

$$t_i = \sup\{t \in [0,1] : (1-t)x_i + ty_i \in K_1\}.$$

Since K_1 and I_i are compact and $y_i \notin K_1$ we get that $t_i \in [0, 1)$. Define

$$\overline{x_i} \stackrel{\text{def}}{=} (1-t_i)x_i + t_i y_i \in K_1.$$

Now set

$$s_i = \inf\{t \in [t_i, 1] : (1-t)x_i + ty_i \in K_2\}.$$

Similarly as above, since K_2 is compact and $\overline{x_i} \notin K_2$, we get that $s_i \in (t_i, 1]$. Define

$$\overline{y_i} \stackrel{\text{def}}{=} (1-s_i)x_i + s_i y_i \in K_2.$$

Further, for $i = 1, 2$, define segments with endpoints $\overline{x_i}, \overline{y_i}$,

$$J_i \stackrel{\text{def}}{=} \{(1-t)\overline{x_i} + t\overline{y_i} : t \in [0,1]\}.$$

Then we get that $J_i \cap K_1 = \{\overline{x_i}\}$, $J_i \cap K_2 = \{\overline{y_i}\}$ and $J_1 \cap J_2 = \emptyset$ (since $J_i \subset I_i$ which were disjoint). Further

$$d(\overline{x_i}, \overline{y_i}) \leq d(x_i, y_i) < \alpha. \tag{3.1}$$

By [9, Theorem 62.V.6] there exists an open connected region G whose boundary is contained in $K_1 \cup K_2 \cup J_1 \cup J_2$. Since $K_1 \cup K_2 \subset f^{-1}(0)$, and by (3.1), we conclude that for all $x \in \mathrm{Bd}(G)$,

$$|f(x)| \leq \Omega(\frac{\alpha}{2}). \tag{3.2}$$

Let $x_0 \in G$ be such that for $i = 1, 2$

$$d(x_0, \overline{x_i}) \geq \frac{1}{2} d(\overline{x_1}, \overline{x_2}).$$

Such a point x_0 exists in G since G is open and connected and thus G is path-connected. By Lemma 3.4 there exists a curve $\phi : [0, \infty) \longrightarrow \mathbb{R}^2$ with Lipschitz constant one, $\phi(0) = x_0$ and $f(\phi(t)) = f(x_0) + t\ sign(f(x_0))$. Since this curve is clearly unbounded, there exists $\tau > 0$ so that $\phi(\tau) \in \mathrm{Bd}(G)$. Then, by (3.2) and since ϕ is Lipschitz with constant one,

$$\begin{aligned}\Omega(\frac{\alpha}{2}) &\geq |f(\phi(\tau))| \geq \tau \geq \|\phi(\tau) - \phi(0)\| = \|\phi(\tau) - x_0\| \\ &\geq d(x_0, J_1 \cup J_2) \geq \min_{i=1,2}(d(x_0, \overline{x_i})) - \alpha \\ &\geq \frac{1}{2} d(\overline{x_1}, \overline{x_2}) - \alpha.\end{aligned}$$

Thus

$$d(\overline{x_1}, \overline{x_2}) \leq 2\Omega(\frac{\alpha}{2}) + 2\alpha,$$

and

$$d(x_1, x_2) \leq d(x_1, \overline{x_1}) + d(\overline{x_1}, \overline{x_2}) + d(\overline{x_2}, x_2) \leq 2\Omega(\frac{\alpha}{2}) + 4\alpha.$$

□

Proposition 3.5. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$, $f^{-1}(t)$ is hereditarily locally connected.*

Proof. Without loss of generality we will assume that f is a co-Lipschitz uniformly continuous map with a co-Lipschitz constant 1, and that $t = 0$. By Theorem 3.2, it is enough to show that $f^{-1}(0)$ does not contain a nontrivial continuum of convergence.

Suppose for contradiction that K_0 is a nontrivial continuum of convergence in $f^{-1}(0)$ and let $x_1, x_2 \in K_0$, and $\beta \stackrel{\text{def}}{=} d(x_1, x_2) > 0$. Let $[K_i]_{i=1}^{\infty}$ be the sequence of mutually disjoint subcontinua of $f^{-1}(0)$ with $\bigcup_i K_i \cap K_0 = \emptyset$ and $\lim[K_i]_i = K_0$. Then, by the definition of the limit (see also [20, Theorem I.(7.2)]), for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ so that for all $x \in K_0$, $d(x, K_n) < \varepsilon$. Thus for $i = 1, 2$ there exists $y_i \in K_n$ with $d(x_i, y_i) < \varepsilon$. Hence, by Lemma 3.3, $d(x_1, x_2) \leq 2\Omega(\varepsilon/2) + 4\varepsilon$. Since ε is arbitrary and $\lim_{r \to 0} \Omega(r) = 0$, we conclude that $d(x_1, x_2) = 0$ which contradicts the fact that K_0 is a nontrivial subcontinuum. ☐

As mentioned in the Introduction, we do not know whether there exist co-Lipschitz uniformly continuous maps or Lipschitz quotient maps from $\mathbb{R}^3$ to $\mathbb{R}$, or in general from $\mathbb{R}^n$ to $\mathbb{R}^k$, with non-locally connected level sets. However we do know that there exist uniform quotient maps from $\mathbb{R}^2$ to $\mathbb{R}$ which have non-locally connected level sets as indicated below.

Example 3.6. *Let $z_n = (\frac{1}{n}, (-1)^n) \in \mathbb{R}^2$ for $n \in \mathbb{Z} \setminus \{0\}$, and let I_n be a segment in $\mathbb{R}^2$ with endpoints z_n, z_{n+1}, when $n > 0$, or z_n, z_{n-1} when $n < 0$. Let I_0 be the vertical segment with endpoints $(0, 1)$ and $(0, -1)$, and let $I_+ = \{(x, -1) : x \geq 1\}$, $I_- = \{(x, -1) : x \leq -1\}$ be two half-lines. Define K to be the union of all these segments $K \stackrel{def}{=} \bigcup_{n \in \mathbb{Z}} I_n \cup I_+ \cup I_-$.*

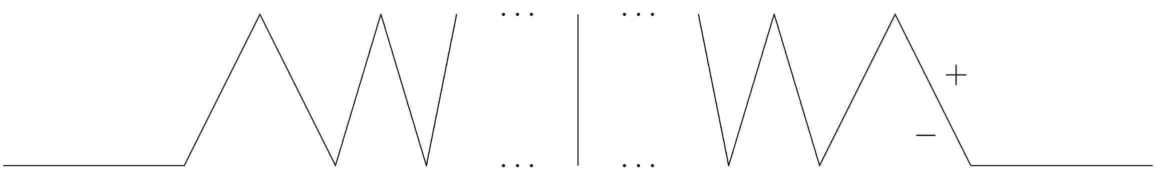


Figure **3.1.** Set K.

The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined as the distance from K multiplied in each component of $\mathbb{R}^2 \setminus K$ by the sign indicated. Then f is Lipschitz and co-uniformly continuous with

$$\omega(r) = \begin{cases} \frac{r^3}{16000} & \text{if } r < \frac{1}{10}, \\ \frac{1}{16 \cdot 10^6} & \text{if } r \geq \frac{1}{10}. \end{cases}$$

Moreover $f^{-1}(0) = K$, which is connected but not locally connected.

The proof of this example is rather tedious and will be published separately.

4. First description of the structure of level sets

In this section we describe the structure of level sets of co-Lipschitz uniformly continuous mappings $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (Theorem 4.11). For that we will need the notions of a dendrite, an order of a point, an end point and a cut point of a topological space M. We recall their definitions below.

Definition 4.1. [20, Chapter III], [9, Chapter VI, §51] Let M be a space and $\mathfrak{n}$ a cardinal number. We say that a point $x \in M$ is of *order* $\leq \mathfrak{n}$ in M provided that for any neighborhood V of x in M, there exists a neighborhood U of x in M with $U \subset V$ and $\text{card}(\text{Bd}(U)) \leq \mathfrak{n}$.

A point $x \in M$ which is of order one in M will be called an *end point* of M.

Definition 4.2. [20, Section III.1], [9, Definition 47.VIII.2] If M is a connected set and p is a point of M such that the set $M \setminus \{p\}$ is not connected, then p is called a *cut point* of M.

Definition 4.3. [20, Section V.1] A continuum M is called a *dendrite* (or an *acyclic curve*) provided that M is locally connected and contains no simple closed curve.

Dendrites constitute a very important class of continua, and they have been extensively studied. We recall here a couple of important properties of dendrites, that we will use.

Theorem 4.4. [20, V.(1.1) and V.(1.2)] *Let M be a continuum. The following statements are equivalent:*
(1) M is a dendrite.
(2) Every point of M is either a cut point or an end point.
(3) M is locally connected and one and only one arc exists between any two points in M.

Our first observation concerning the structure of level sets of co-Lipschitz uniformly continuous mappings is the following:

Corollary 4.5. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$ and every subcontinuum M of $f^{-1}(t)$, M is a dendrite.*

Proof. It is easy to see that when f is a co-Lipschitz uniformly continuous map then for all t, $f^{-1}(t)$ cannot contain a simple closed curve. Indeed, since f is co-Lipschitz, $f^{-1}(t)$ has empty interior and if a simple closed curve C was contained in $f^{-1}(t)$ then $\mathbb{R}^2 \setminus f^{-1}(t)$ would have a bounded component A contained in the region inside the curve C. But then $f(A) = (t, \infty)$ or $(-\infty, t)$, which is impossible since $\overline{A}$ is compact and f is continuous.

Further, by Proposition 3.5, every subcontinuum M of $f^{-1}(t)$ is locally connected. Thus M is a dendrite. □

Our next goal is to show that every $f^{-1}(t)$ is of a particularly simple form, that every point of $f^{-1}(t)$ is of finite order and only finitely many points in $f^{-1}(t)$ have order bigger than 2. Thus we will show that $f^{-1}(t)$ has a tree structure. We start from the following:

Proposition 4.6. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and let K be a component of $f^{-1}(t)$ for some $t \in \mathbb{R}$. Then every point of K is a cut point, in particular $f^{-1}(t)$ has no end points.*

Proof. Suppose that $x \in K$ is not a cut point of K. We first show that this implies that x is an endpoint of K. Assume for contradiction that x is not an endpoint of K. By Proposition 3.5, there exists a neighborhood U of x so that U is bounded and $U \cap K$ is connected. Then x is not an endpoint of $\overline{U} \cap K$. Since, by Corollary 4.5, $\overline{U} \cap K$ is a dendrite, we conclude by Theorem 4.4, that x is a cut point of $\overline{U} \cap K$. Let y, z be two points in different components of $\overline{U} \cap K \setminus \{x\}$. Then x belongs to the unique path which connects y and z in $\overline{U} \cap K$.

On the other hand, by [20, III,(4.15)], there exists a neighborhood U_1 of x so that $K \setminus U_1$ is connected and $y, z \in K \setminus U_1$. Then, by [20, I,(8.2)], there exists a subcontinuum M of $K \setminus U_1$ so that $y, z \in M$. Thus by Corollary 4.5 and Theorem 4.4, there exists a path in M which connects y and z. Thus we conclude that in $(\overline{U} \cap K) \cup M$ there exist two different paths connecting y and z, which contradicts the fact that $(\overline{U} \cap K) \cup M$ is a dendrite. Hence x is an endpoint of K.

It follows from [19, Theorem 26] (cf. also [5, Proof of Theorem 27], where this fact is attributed to R.G. Lubben), that if $x \in K$ is an endpoint of K then x belongs to the boundary of exactly one component of $\mathbb{R}^2 \setminus K$. But then, since $\mathbb{R}^2 \setminus f^{-1}(t)$ and thus also $\mathbb{R}^2 \setminus K$ have finite number of components, there exists a neighborhood U of x so that U

intersects exactly one component of $\mathbb{R}^2 \setminus K$. Hence $f(U) \subset (t,\infty)$ or $f(U) \subset (-\infty,t)$, which contradicts the fact that f is co-Lipschitz. □

For the analysis of the structure of level sets $f^{-1}(t)$ we will need the following notion introduced by Shimrat [15, 16]. Below the term "countable" includes finite (and also empty).

Definition 4.7. [16] A set K is called a *ramification* (Stone [18] uses the term *open ramification*) if K is a finite or countable union of the form

$$K = p_0 \cup \bigcup_i R_i \cup \bigcup_{ij} R_{ij} \cup \cdots \cup \bigcup_{ij\ldots m} R_{ij\ldots m} \cup \ldots \tag{4.1}$$

where p_0 is a point; R_i are (at least two) half-open arcs (i.e. homeomorphic images of $[0,\infty)$) having p_0 as a common endpoint, but otherwise disjoint; every $R_{ij\ldots mn}$ is a half-open arc with an endpoint on $R_{ij\ldots m}$. An intersection between any two half-open arcs is an endpoint of at least one of them. The number of half-open arcs having endpoints on a definite $R_{ij\ldots m}$ is finite (≥ 0 and ≥ 2 in the case of the R_i) or countable and their endpoints need not be distinct. K is given its "length-metric", i.e., each half-open arc is metrized as a linear interval, and the distance between two points $x,y \in K$ is the sum of the lengths of the intervals making up the unique path from x to y in K.

Shimrat [16] proved the following characterization of ramifications.

Theorem 4.8. [16, Theorem 3] *A connected set K is a ramification if and only if K is locally connected and every point of K is a cut point.*

As an immediate corollary of Theorem 4.8 and Propositions 3.5 and 4.6 we obtain:

Corollary 4.9. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and let K be a component of $f^{-1}(t)$ for some $t \in \mathbb{R}$. Then K is a ramification.*

Our next goal is show that components of level sets are finite ramifications. To precisely analyze their form it will be convenient to use an algebraic topology notion of a number of ends of an unbounded locally connected set.

Definition 4.10. [6, Definition 1.18] We say that a connected locally connected Hausdorff space W *has at least k ends* if there exists an open subspace $V \subseteq W$ with compact closure $\overline{V}$ so that $W \setminus \overline{V}$ has at least k unbounded components. The space W *has exactly k ends* (denoted $\#e(W) = k$) if W has at least k ends but not at least $k+1$ ends.

One should be careful not to confuse ends with end points. We think of ends, intuitively, as infinite ends of unbounded sets. In fact, there exist ways of making this intuition precise, by defining ends using homotopy classes of unbounded paths contained in the space W (see [6]), but we will not need this for our present purpose.

Clearly continua never have any ends, but unbounded locally connected sets may have some end points in addition to the fact that they always have at least one end.

If a locally connected space W has a finite number of connected components, $W = \bigcup_{j=1}^m C_j$, then we will use notation $\#e(W)$ to mean the sum of $\#e(C_j)$, i.e.

$$\#e(W) \stackrel{\text{def}}{=} \sum_{j=1}^{m} \#e(C_j).$$

We are now ready to state the main result of this section which describes the structure of level sets $f^{-1}(t)$.

Theorem 4.11. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$, every component K of $f^{-1}(t)$ has a representation of the form:*

$$K = K_0 \cup \bigcup_{j=1}^{n} K_j,$$

where $n \in \mathbb{N}$, $n = \#e(K)$, K_0 is a compact connected tree (i.e. a graph containing no closed curve) with exactly n endpoints, each K_j is a ray, that is a closed unbounded set homeomorphic with $[0, \infty)$, sets $\{K_j\}_{j=1}^n$ are mutually disjoint, for all j, $\operatorname{card}(K_j \cap K_0) = 1$ and the unique point in the intersection $K_j \cap K_0$ is an end point of K_0 and of K_j.

Moreover n is equal to the number of components of $\mathbb{R}^n \setminus K$.

Definition 4.12. We will use the term *unbounded finite tree* for sets of the form described in Theorem 4.11.

Proof. We note first that since K is closed and K has form (4.1), thus all half-open arcs in $R_{ij\ldots m}$ are unbounded, i.e. they all are rays. Further since, by Theorem 1.1, the number of components of $\mathbb{R}^n \setminus K$ is finite, we conclude that the total number of rays in (4.1) is finite and equal to both the number of ends of K and the number of components of $\mathbb{R}^n \setminus K$.

To construct sets $\{K_j\}_{j=0}^n$, let, for $j = 1, \ldots, n$, x_j be any point other than the endpoint in each of the n rays $\{r_j\}_{j=1}^n$ of K in (4.1), and let K_j be the union of $\{x_j\}$ and the undbounded component of $r_j \setminus \{x_j\}$. Then, by (4.1), sets $\{K_j\}_{j=1}^n$ are mutually disjoint. Define K_0 as the union of unique paths connecting each two of the points from $\{x_j\}_{j=1}^n$ in K (alternatively, $K_0 = (K \setminus \bigcup_{j=1}^n K_j) \cup \{x_j\}_{j=1}^n$). Then for each $j = 1, \ldots, n$, $K_0 \cap K_j = \{x_j\}$ and the set $\{x_j\}_{j=1}^n$ is equal to the set of all endpoints of K_0, and every other point of K_0 is a cut point of K_0. It follows that K_0 is a compact tree with exactly n endpoints $\{x_j\}_{j=1}^n$. This can be seen either directly from the form of ramification (4.1), or one can use a stronger result of Stone [18] (cf. [13], [12, Theorem 9.24]) which characterizes finite graphs as continua with the property that they are disconnected by a removal of any subset of points of fixed finite cardinality. ☐

5. Number and form of components of level sets

In this section we present an exact characterization of the form of level sets of a co-Lipschitz uniformly continuous map f from $\mathbb{R}^2$ to $\mathbb{R}$ (Theorem 5.1), which significantly refines Theorem 4.11. In particular, we obtain an affirmative answer to the question posed in [7] whether the number of components of level sets $f^{-1}(t)$ or of $\mathbb{R}^2 \setminus f^{-1}(t)$ are constant after excluding finitely many values of t. We begin with the statement of our main characterization theorem.

We will use the notation $\#c(W)$ to denote the number of components of the set W.

Theorem 5.1. *For every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r \to 0} \Omega(r) = 0$, there exists a natural number $M = M(c, \Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω, there exists a natural number $n = n(f) \le M$ and a finite subset T_f of $\mathbb{R}$, with $\operatorname{card}(T_f) \le n - 1$, so that:*

(1) *for all $t \in \mathbb{R}$,*
   (a) *$\#e(f^{-1}(t)) = 2n$, that is there exists $R_0 \in \mathbb{R}$ so that for every $R > R_0$, $f^{-1}(t) \setminus B(0, R)$ has exactly $2n$ unbounded components.*

(b) *If* $\{C_{R,i}\}_{i=1}^{2n}$ *are the unbounded components of* $f^{-1}(t)\setminus B(0,R)$*, then for all* $i \neq j$*,* $\lim_{R\to\infty} d(C_{R,i}, C_{R,j}) = \infty$;

(2) *for all* $t \in \mathbb{R}\setminus T_f$*,*
   (a) $\#c(f^{-1}(t)) = n$,
   (b) $\#c(\mathbb{R}^2 \setminus f^{-1}(t)) = n+1$,
   (c) *each component of* $f^{-1}(t)$ *is homeomorphic with the real line and separates the plane into exactly 2 components;*

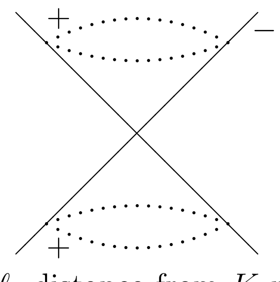
(3) *for all* $t_i \in T_f$*,*
   (a) $\#c(f^{-1}(t_i)) < n$,
   (b) $\#c(\mathbb{R}^2 \setminus f^{-1}(t_i)) = 2n+1-\#c(f^{-1}(t_i)) \in (n+1, 2n)$,
   (c) *each component of* $f^{-1}(t_i)$ *is an unbounded finite tree, i.e. has the form described in Theorem 4.11*

Remark 5.2. Theorem 5.1 is analogous to a result of Johnson, Lindenstrauss, Preiss and Schechtman [7], who proved that for every pair of a constant $c > 0$ and a function $\Omega(\cdot)$ with $\lim_{r\to 0}\Omega(r) = 0$, there exists a natural number $M = M(c,\Omega)$, so that for every co-Lipschitz uniformly continuous map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with a co-Lipschitz constant c and a modulus of uniform continuity Ω, there exists a natural number $n = n(f) \le M$ and a polynomial P_f with degree equal to n, so that $f = P_f \circ h_f$, where h_f is a homeomorphism of $\mathbb{R}^2$. Hence there exists a finite set $T_f \subset \mathbb{R}^2$ with $\mathrm{card}(T_f) \le n \le M$, so that for all $t \in \mathbb{R}^2 \setminus T_f$, $\mathrm{card}(f^{-1}(t)) = n$ and for all $t_i \in T_f$, $\mathrm{card}(f^{-1}(t_i)) < n$, analogously with parts (2a) and (3a) of Theorem 5.1.

For Lipschitz quotient maps from $\mathbb{R}^2$ to $\mathbb{R}^2$, Maleva [11] studied the dependence of the number $M(c,L)$ on the Lipschitz and co-Lipschitz constants L and c. Maleva proved in particular that there exists a scale $0 < \cdots < \varrho_2^{(m)} < \cdots < \varrho_2^{(1)} < 1$ such that for any Lipschitz quotient mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the condition $c/L > \varrho_2^{(m)}$ implies that $\mathrm{card}(f^{-1}(t)) \le m$ for all $t \in \mathbb{R}^2$ (in fact this holds with $\varrho_2^{(m)} = 1/(m+1)$) [11, Theorem 2].

It is natural to ask whether a similar scale exists for the numbers $M(c,\Omega)$ defined in Theorem 5.1. [After reading a preliminary version of this paper, Maleva proved the existence of such a scale. Namely she proved that if $c/L > \sin(\pi/(2n))$ then for all $t \in \mathbb{R}$, $\#c(f^{-1}(t)) < n$, [10]. A similar scale also exists for co-Lipschitz uniformly continuous maps [10].]

Remark 5.3. Theorem 5.1 does not generalize for Lipschitz quotient mappings $f : \mathbb{R}^m \longrightarrow \mathbb{R}$, for $m > 2$. Indeed, consider a surface K in $\mathbb{R}^3$ obtained by rotation about the $z-$axis of the line $\{(x,0,x) : x \in \mathbb{R}\}$.



Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the ℓ_1-distance from K multiplied in each component of $\mathbb{R}^3 \setminus K$ by the sign indicated. Then f is a Lipschitz quotient map and for all $t > 0$, $\#c(f^{-1}(t)) = 2$ and for all $t \le 0$, $\#c(f^{-1}(t)) = 1$. It is easy to construct analogous examples for arbitrary $m \ge 3$ and for many finite arrangements of $\#c(f^{-1}(t))$ in different subintervals of $\mathbb{R}$.

Remark 5.4. The estimate of the cardinality of the exceptional set T_f is best possible, in the sense that for any $n \in \mathbb{N}$ it is easy to construct examples of Lipschitz quotient mappings $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ so that $\text{card}(T_f) = n - 1$ and $\#c(f^{-1}(t)) = n$ for all $t \in \mathbb{R} \setminus T_f$. In Figure 5.1 below, we present sketches of examples of such functions for $n = 2, 3, 4$. In each sketch, level sets for different values of t are represented by different styles of lines (within limits set by the drawing program (XY-pic)), and the mapping f is the distance in the ℓ_1-metric from the solid lines, which represent the preimage of 0, multiplied, in each component of the complement of the solid lines, by the sign indicated.

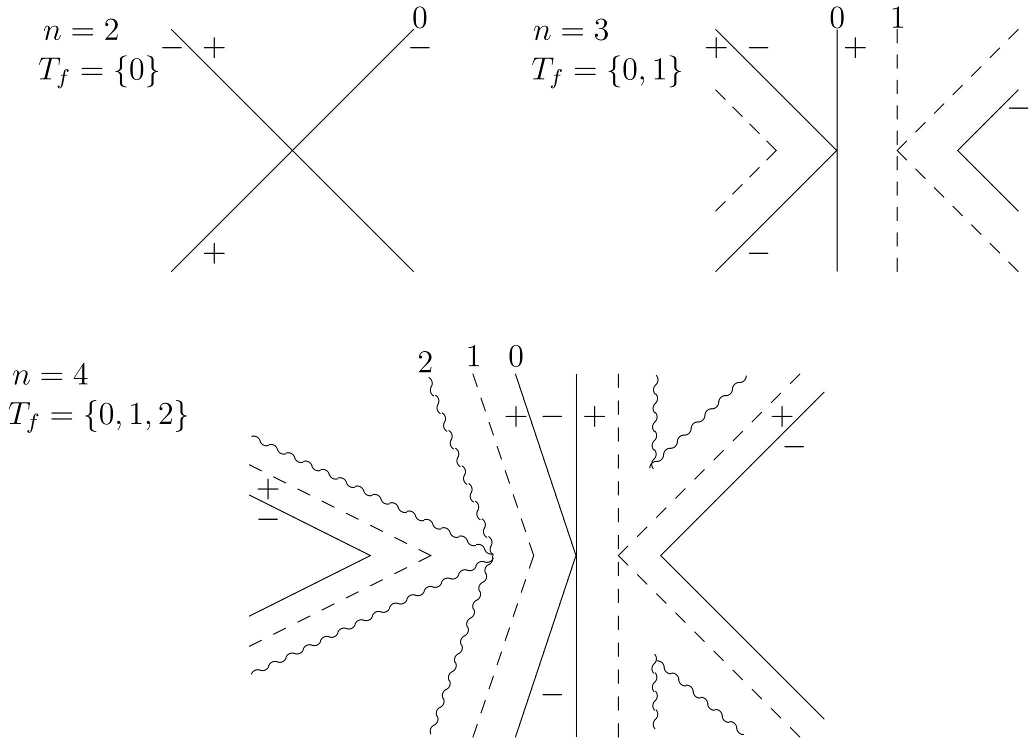


FIGURE **5.1.**

Remark 5.5. After reading a preliminary version of this paper, Maleva has strengthened the conclusion of Theorem 5.1(1b). She proved [10], in the notation as above, that there exists a constant $\delta > 0$ depending only on the modulus of continuity of f and its co-Lipschitz constant, so that for every $t \in \mathbb{R}$ there exists $R(t) > 0$ so that for all $R > R(t)$ and all $i \neq j$, $d(C_{R,i}, C_{R,j}) \geq \delta R$. This has consequences not only for the topology, but also for the allowable geometric structure of $f^{-1}(t)$, e.g. $f^{-1}(t)$ cannot contain a parabola, see [10].

For the proof of Theorem 5.1 we will need a large number of auxiliary results concerning the number of components of level sets $f^{-1}(t)$ and the end structure of boundaries of components of the complements of $f^{-1}(t)$. We start from a presentation of these results and postpone the proof of Theorem 5.1 to the end of this section.

Our first observations analyze the number of ends of boundaries of components of $\mathbb{R}^2 \setminus f^{-1}(t)$.

Proposition 5.6. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and K be a connected component of the level set $f^{-1}(t)$. Then each component of $\mathbb{R} \setminus K$ is homeomorphic with $\mathbb{R}^2$ and its boundary is connected and has exactly 2 ends.*

Moreover, $\#c(\mathbb{R}^2 \setminus K) = \#e(K)$. In particular, $\#e(K) \geq 2$.

Proof. By [9, Theorem 61.II.4], the boundary of every component of $\mathcal{S}_2 \setminus \overline{K}$ is a simple closed curve. Thus by [9, Theorem 61.V.1] each component of $\mathcal{S}_2 \setminus \overline{K}$, and therefore also of $\mathbb{R}^2 \setminus K$, is homeomorphic with $\mathbb{R}^2$. Since ∞ belongs to the boundary of every component of $\mathcal{S}_2 \setminus \overline{K}$, and since this boundary is a simple closed curve, we conclude that the order of ∞, as a point of the boundary of any component of $\mathcal{S}_2 \setminus \overline{K}$, is equal to two and thus this boundary has exactly 2 ends, and it is connected as a subset of $\mathbb{R}^2$.

The moreover part follows directly from Theorem 4.11. □

Propostion 5.6 has two useful consequences.

Corollary 5.7. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and A be a component of $\mathbb{R}^2 \setminus f^{-1}(t)$. Then*

$$\#e(\mathrm{Bd}(A)) = 2\#c(f^{-1}(t) \cap \mathrm{Bd}(A)).$$

Proof. Let $\{K_i\}_{i=1}^n$ be components of $f^{-1}(t)$. If $\#c(f^{-1}(t) \cap \mathrm{Bd}(A)) = 1$, then A is a component of, say, $\mathbb{R}^2 \setminus K_1$ and, by Proposition 5.6, $\#e(\mathrm{Bd}(A)) = 2$.

We suppose, for the induction, that if $\#c(f^{-1}(t) \cap \mathrm{Bd}(A)) \leq k$, i.e. if A is a component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^k K_j$, then

$$\#e(\mathrm{Bd}(A)) = 2\#c(f^{-1}(t) \cap \mathrm{Bd}(A)). \tag{5.1}$$

Let B be a component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^{k+1} K_j$, say $B = A \cap C$, where A is a component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^k K_j$ and C is a component of $\mathbb{R}^2 \setminus K_{k+1}$.

If $K_{k+1} \not\subset A$, then, by the connectedness of $K_{k+1}, K_{k+1} \cap A = \emptyset$ and either $A \subset C$ or $A \cap C = \emptyset$. Since $B \neq \emptyset$, we obtain that $B = A$ and B is a component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^k K_j$ and, by the inductive hypothesis, there is nothing to prove.

Thus, without loss of generality, we assume that $K_{k+1} \subset A$ and $\mathrm{Bd}(C) \subset K_{k+1} \subset A$. Then

$$\mathrm{Bd}(C) \subset \mathrm{Bd}(A \cap C). \tag{5.2}$$

Similarly, if $\bigcup_{j=1}^k K_j \not\subset C$ then by the connectedness of sets $\{K_j\}_{j=1}^k$, there exists $i_0 \leq k$ so that $K_{i_0} \not\subset C$ and thus $K_{i_0} \cap C = \emptyset$. Hence any component of $\mathbb{R}^2 \setminus K_{i_0}$ is either disjoint with C, or contains C. Thus B can be represented as an intersection of components of $\{\mathbb{R}^2 \setminus K_j\}_{j=1, j\neq i_0}^{k+1}$ and, by the inductive hypothesis, we are done.

Thus, without loss of generality, we assume that $\bigcup_{j=1}^k K_j \subset C$. Hence, as before, $\mathrm{Bd}(A) \subset \bigcup_{j=1}^k K_j \subset C$ and

$$\mathrm{Bd}(A) \subset \mathrm{Bd}(A \cap C). \tag{5.3}$$

By [8, Formula 6.II(8)], we have

$$\mathrm{Bd}(A \cap C) \subset \mathrm{Bd}(A) \cup \mathrm{Bd}(C). \tag{5.4}$$

Combining (5.2), (5.3) and (5.4) we get $\mathrm{Bd}(A \cap C) = \mathrm{Bd}(A) \cup \mathrm{Bd}(C)$, and, since $\mathrm{Bd}(A) \cap \mathrm{Bd}(C) = \emptyset$, we conclude that $\#e(\mathrm{Bd}(A \cap C)) = \#e(\mathrm{Bd}(A)) + \#e(\mathrm{Bd}(C))$. Thus, by (5.1) and (5.2),

$$\begin{aligned} \#e(\mathrm{Bd}(B)) &= 2\#c(f^{-1}(t) \cap \mathrm{Bd}(A)) + 2 \\ &= 2\#c(f^{-1}(t) \cap (\mathrm{Bd}(A) \cup K_{k+1})) \\ &= 2\#c(f^{-1}(t) \cap \mathrm{Bd}(B)), \end{aligned}$$

which ends the proof. □

Corollary 5.8. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and $t \in \mathbb{R}$. Then*

$$\#c(f^{-1}(t)) + \#c(\mathbb{R}^2 \setminus f^{-1}(t)) = \#e(f^{-1}(t)) + 1. \tag{5.5}$$

Proof. By Theorems 1.1 and 2.4 both $\mathbb{R}^2 \setminus f^{-1}(t)$ and $f^{-1}(t)$ have finite number of components. Denote $l = \#c(f^{-1}(t))$ and let $\{K_j\}_{j=1}^l$ be the components of $f^{-1}(t)$. If $l = 1$, (5.5) follows directly from Proposition 5.6. If $l > 1$ then K_2 is contained in exactly one of the components of $\mathbb{R}^2 \setminus K_1$, say C. Since C is homeomorphic with $\mathbb{R}^2$, again by Proposition 5.6 we conclude that $\#c(C \setminus K_2) = \#e(K_2)$, and thus

$$\begin{aligned} 2 + \#c(\mathbb{R}^2 \setminus (K_1 \cup K_2)) &= 2 + \#c(\mathbb{R}^2 \setminus K_1) - 1 + \#c(\mathbb{R}^2 \setminus K_2) \\ &= \#e(K_1) + \#e(K_2) + 1. \end{aligned}$$

By induction we obtain (5.5) for any $l \in \mathbb{N}$. □

As a consequence of Theorems 1.1, 2.4 and Corollary 5.8 we immediately obtain:

Corollary 5.9. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map. Then for every $t \in \mathbb{R}$, the number of ends of $f^{-1}(t)$ is finite and bounded by a function depending only on the co-Lipschitz constant of f and its modulus of uniform continuity.*

Our next goal is to show that the number of ends of $f^{-1}(t)$ is independent of t. To achieve this we first prove that different ends of level sets $f^{-1}(t)$ are "infinitely far away" from each other, as on Figure 1.2 in the Introduction. To state this precisely we will use the notation $d(X, Y)$ to denote the distance between sets X, Y i.e.

$$d(X, Y) \stackrel{\text{def}}{=} \inf\{d(x, y) : x \in X, y \in Y\}.$$

Proposition 5.10. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and $t \in \mathbb{R}$. If $f^{-1}(t)$ has l components $\{K^i(t)\}_{i=1}^l$, and each component $K^i(t)$ has the following representation of the form described in Theorem 4.11,*

$$K^i(t) = K_0^i(t) \cup \bigcup_{j=1}^{n(i)} K_j^i(t),$$

then, for all $i_1, i_2 \in \{1, \cdots, l\}, j_1 \in \{1, \cdots n(i_1)\}, j_2 \in \{1, \cdots, n(i_2)\}$, if the ordered pairs $(i_1, j_1), (i_2, j_2)$ are not the same, then

$$\lim_{R \to \infty} d(K_{j_1}^{i_1}(t) \setminus B(0, R), K_{j_2}^{i_2}(t) \setminus B(0, R)) = \infty.$$

Proof. This result follows almost immediately from Lemma 3.3. Without loss of generality we assume that the co-Lipschitz constant of f is 1 and let $\Omega(\cdot)$ be the modulus of uniform continuity of f. For any $R \in \mathbb{R}_+$ denote

$$d_R = d(K_{j_1}^{i_1}(t) \setminus B(0, R), K_{j_2}^{i_2}(t) \setminus B(0, R)).$$

Clearly $d_{R_1} \geq d_{R_2}$ when $R_1 \geq R_2$, thus, if $\lim_{R \to \infty} d_R \neq \infty$ then there exists $\alpha \in \mathbb{R}_+$ so that for all $R \in \mathbb{R}$,

$$d_R \leq \alpha. \tag{5.6}$$

Fix $x_1 \in K_{j_1}^{i_1}(t)$ and $y_1 \in K_{j_2}^{i_2}(t)$ so that $d(x_1, y_1) \leq \alpha$. Set

$$\tilde{R} \stackrel{\text{def}}{=} \|x_1\| + 2\Omega(\frac{\alpha}{2}) + 4\alpha + 1.$$

Then, by (5.6), there exist $x_2 \in K^{i_1}_{j_1}(t) \setminus B(0,\tilde{R})$ and $y_2 \in K^{i_2}_{j_2}(t) \setminus B(0,\tilde{R})$ with $d(x_2, y_2) \le \alpha$. Since, for $\nu = 1, 2$, the sets $K^{i_\nu}_{j_\nu}(t)$ are connected subsets of $f^{-1}(t)$, there exist arcs $\sigma_\nu \subset K^{i_\nu}_{j_\nu}(t) \subset f^{-1}(t)$ with endpoints x_ν, y_ν. Since $(i_1, j_1) \neq (i_2, j_2)$, the arcs $\sigma_\nu, \nu = 1, 2$, are disjoint subcontinua of $f^{-1}(t)$, and hence, by Lemma 3.3,

$$d(x_1, x_2) \le 2\Omega(\frac{\alpha}{2}) + 4\alpha.$$

But

$$d(x_1, x_2) \ge \big|\|x_2\| - \|x_1\|\big| \ge \tilde{R} - \|x_1\| = 2\Omega(\frac{\alpha}{2}) + 4\alpha + 1,$$

and the resulting contradiction ends the proof of Proposition 5.10. □

As an immediate corollary we obtain the following two facts which we state here for an easy reference.

Corollary 5.11. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and $f^{-1}(t)$ have l components $\{K^i(t)\}_{i=1}^l$. Then for all $i_1, i_2 \le l$, $i_1 \neq i_2$, we have*

$$d(K^{i_1}(t), K^{i_2}(t)) > 0.$$

Corollary 5.12. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$, $R \in \mathbb{R}_+$ and $\{C_{R,i}\}_i$ are a collection of unbounded components of $f^{-1}(t) \setminus B(0, R)$. Then, for all $i \neq j$,*

$$\lim_{R\to\infty} d(C_{R,i}, C_{R,j}) = \infty.$$

As a consequence of Proposition 5.10, we obtain three somewhat technical facts which will be important for our further arguments.

Lemma 5.13. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map and $t_1, t_2 \in \mathbb{R}$. Then*

(a) $\#e(f^{-1}(t_1)) = \#e(f^{-1}(t_2))$.

(b) *If $t_1 > t_2$, A is a component of $f^{-1}(t_2, \infty)$ or $A = \mathbb{R}^2$ and $f^{-1}(t_1, \infty) \cap A = \bigcup_{\lambda=1}^n A_\lambda$, where A_λ are components of $f^{-1}(t_1, \infty) \cap A$, then*

$$\sum_{\lambda=1}^{n} \#e(\mathrm{Bd}(A_\lambda)) = \#e(f^{-1}(t_1) \cap A).$$

(c) *If $t_1 > t_2$ and A is a component of $f^{-1}(t_2, \infty)$, then*

$$\#e(f^{-1}(t_1) \cap A) = \#e(\mathrm{Bd}(A)).$$

Proof. As before we assume without loss of generality that the co-Lipschitz constant of f is 1. By Corollary 5.8 (or a combination of earlier results in this paper) we know that both $f^{-1}(t_1)$ and $f^{-1}(t_2)$ have a finite number of ends. By Theorems 2.4 and 4.11, for $\nu = 1, 2$, $f^{-1}(t_\nu)$ can be represented as

$$f^{-1}(t_\nu) = \bigcup_{i=1}^{l_\nu} \Big(K_0^i(t_\nu) \cup \bigcup_{j=1}^{n_\nu(i)} K_j^i(t_\nu)\Big), \tag{5.7}$$

where $l_\nu, n_\nu(i) \in \mathbb{N}$ and $K^i_j(t_\nu)$ are mutually disjoint unbounded rays in $\mathbb{R}^2$, and sets $K^i_0(t_\nu)$ are compact. Further, for $\nu = 1, 2$, the number of distinct rays $K^i_j(t_\nu)$ is finite and equals the number of ends of $f^{-1}(t_\nu)$, i.e.

$$\operatorname{card} \mathcal{R}(t_\nu) = \#e(f^{-1}(t_\nu)), \tag{5.8}$$

where $\mathcal{R}(t_\nu) \stackrel{\text{def}}{=} \{K_j^i(t_\nu) : i = 1, \ldots, l_1, j = 1, \ldots, n_\nu(i)\}$. By Proposition 5.10, if Y_1, Y_2 are distinct rays of the same level set $f^{-1}(t_\nu)$, where $\nu \in \{1, 2\}$, then $\lim_{R\to\infty} d(Y_1 \setminus B(0, R), Y_2 \setminus B(0, R)) = \infty$. Thus, there exists $R_0 \in \mathbb{R}$ so that, for $\nu = 1, 2$, and for all $i \leq l_\nu$,

$$K_0^i(t_\nu) \subset B(0, R_0 - 1), \tag{5.9}$$

and so that, for any distinct rays Y_1, Y_2 of the same level set $f^{-1}(t_\nu)$, where $\nu = 1$ or 2,

$$d(Y_1 \setminus B(0, R_0), Y_2 \setminus B(0, R_0)) \geq 1 + 4|t_1 - t_2|. \tag{5.10}$$

On the other hand, since f is co-Lipschitz with constant 1, for every $x \in f^{-1}(t_1)$ and for every $r > 0$,

$$B(f(x), r) = (t_1 - r, t_1 + r) \subset f(B(x, r)).$$

Since $t_2 \in (t_1 - r, t_1 + r)$ when $r = 2|t_1 - t_2|$, we conclude that

$$\text{for every } x \in f^{-1}(t_1) \text{ there exists } y \in B(x, 2|t_1 - t_2|) \cap f^{-1}(t_2). \tag{5.11}$$

Now let $X \in \mathcal{R}(t_1)$ be a ray from the representation of $f^{-1}(t_1)$ described in (5.7), and let $x \in X$ be such that $\|x\| \geq R_0 + 2|t_1 - t_2|$. Then by (5.11) and (5.9) there exists at least one ray $Y \in \mathcal{R}(t_2)$ so that $d(x, Y \setminus B(0, R_0)) < 2|t_1 - t_2|$.

Suppose that there exist two distinct rays $Y_1, Y_2 \in \mathcal{R}(t_2)$ so that for $\alpha = 1, 2$,

$$d(x, Y_\alpha \setminus B(0, R_0)) < 2|t_1 - t_2|.$$

But then we would have

$$d(Y_1 \setminus B(0, R_0), Y_2 \setminus B(0, R_0)) < 4|t_1 - t_2|,$$

which contradicts (5.10). Additionally, for every X the ray Y is determined uniquely since the distance between X and Y is strictly less than $2|t_1 - t_2|$.

Thus we have described a one-to-one map γ from the set of rays of $f^{-1}(t_1)$ into the set of rays of $f^{-1}(t_2)$, i.e. from $\mathcal{R}(t_1)$ into $\mathcal{R}(t_2)$, and γ operates in such a way that for every $X \in \mathcal{R}(t_1)$ and for every $x \in X$ with $\|x\| \geq R_0 + 2|t_1 - t_2|$ we have

$$d(x, \gamma(X) \setminus B(0, R_0)) < 2|t_1 - t_2|. \tag{5.12}$$

Since γ is one-to-one, by (5.8) and by symmetry, we have $\#e(f^{-1}(t_1)) = \#e(f^{-1}(t_2))$, which ends the proof of part (a).

Moreover, we conclude that γ is a bijection from $\mathcal{R}(t_1)$ onto $\mathcal{R}(t_2)$.

To prove part (b) we keep the same notation as above and we note that if $f^{-1}(t_1, \infty) \cap A = \bigcup_{\lambda=1}^n A_\lambda$, where A_λ are components of $f^{-1}(t_1, \infty) \cap A$, then

$$f^{-1}(t_1) \cap A = \bigcup_{\lambda=1}^{n} \mathrm{Bd}(A_\lambda), \tag{5.13}$$

and therefore

$$\#e(f^{-1}(t_1) \cap A) = \#e\Big(\bigcup_{\lambda=1}^{n} \mathrm{Bd}(A_\lambda)\Big).$$

Denote by $\mathcal{R}_A(t_1)$ the set of rays of $f^{-1}(t_1)$ contained in A, i.e.

$$\mathcal{R}_A(t_1) \stackrel{\text{def}}{=} \{K_j^i(t_1) \in \mathcal{R}(t_1) : K_j^i(t_1) \subset A\}.$$

Since each ray $K_j^i(t_1)$ has exactly one end, we get

$$\#e(f^{-1}(t_1) \cap A) = \mathrm{card}(\mathcal{R}_A(t_1)). \tag{5.14}$$

By Theorem 4.11, for $R_0 \in \mathbb{R}_+$ defined above, and for each $K_j^i(t_1) \in \mathcal{R}_A(t_1)$, the set $K_j^i(t_1) \setminus B(0, R_0)$ has a unique unbounded component; we will denote these components by $\{X_\alpha : \alpha = 1, \ldots, \#e(f^{-1}(t_1) \cap A)\}$. Note that, by Theorem 4.11, each X_α is homeomorphic with $[0, \infty)$. We will show that

(5.15) for each $\alpha \leq \#e(f^{-1}(t_1) \cap A)$ there exists a unique $\lambda(\alpha) \leq n$ with $X_\alpha \subset \mathrm{Bd}(A_{\lambda(\alpha)})$.

Once (5.15) is established, part (b) follows easily. Indeed, by (5.15) we can define sets

$$E_\lambda \stackrel{\text{def}}{=} \{\alpha \leq \#e(f^{-1}(t_1) \cap A) : X_\alpha \subset \mathrm{Bd}(A_\lambda)\},$$

and sets E_λ are disjoint. Note that $\mathrm{card}(E_\lambda) = \#e(\mathrm{Bd}(A_\lambda))$. Moreover, by (5.13),

$$\bigcup_{\lambda=1}^{n} \mathrm{Bd}(A_\lambda) \supset \{X_\alpha : \alpha = 1, \ldots, \#e(f^{-1}(t_1) \cap A)\},$$

so $\bigcup_{\lambda=1}^{n} E_\lambda = \{1, \ldots, \#e(f^{-1}(t_1) \cap A)\}$ and thus $\#e(f^{-1}(t_1) \cap A) = \sum_{\lambda=1}^{n} \mathrm{card}(E_\lambda) = \sum_{\lambda=1}^{n} \#e(\mathrm{Bd}(A_\lambda))$, as desired.

To prove (5.15), note that by (5.13) and since sets $\mathrm{Bd}(A_\alpha)$ are closed, for each $\alpha \leq \#e(f^{-1}(t_1) \cap A)$:

$$\overline{X_\alpha} \subset \bigcup_{\lambda=1}^{n} \mathrm{Bd}(A_\lambda).$$

Thus

$$\overline{X_\alpha} = \bigcup_{\lambda=1}^{n} (\overline{X_\alpha} \cap \mathrm{Bd}(A_\lambda)).$$

Since $\overline{X_\alpha}$ is connected and sets $\overline{X_\alpha} \cap \mathrm{Bd}(A_\lambda)$ are closed, we conclude that either there exists a unique $\lambda(\alpha)$ so that, for all $\lambda \neq \lambda(\alpha)$, $\overline{X_\alpha} \cap \mathrm{Bd}(A_\lambda) = \emptyset$, and in this case part (b) holds, or otherwise there exist $\lambda_1, \lambda_2 \leq n, \lambda_1 \neq \lambda_2$ so that

$$(\overline{X_\alpha} \cap \mathrm{Bd}(A_{\lambda_1})) \cap (\overline{X_\alpha} \cap \mathrm{Bd}(A_{\lambda_2})) \neq \emptyset. \tag{5.16}$$

But this alternative leads to a contradiction. Indeed, suppose that $x \in \overline{X_\alpha} \cap \mathrm{Bd}(A_{\lambda_1}) \cap \mathrm{Bd}(A_{\lambda_2})$. Since $\overline{X_\alpha}$ is a ray, i.e. a homeomorph of $[0, \infty)$, contained in one of the rays $\{K_j^i(t_1)\}_{i,j}$ of $f^{-1}(t_1) \cap A$, and since, by (5.9), $\overline{X_\alpha}$ is disjoint with all sets $\{K_0^i(t_1)\}_i$ we conclude that the order of the point x in $f^{-1}(t_1)$ is equal to 2. Hence, by Definition 4.1, for every neighborhood V of x, there exists a neighborhood of x with $U \subset V$ and so that $\mathrm{card}(\mathrm{Bd}(U) \cap f^{-1}(t_1)) = 2$. By (5.10) and since $f^{-1}(t_1)$ is locally connected, we can choose $U \subset A$ so that $x \in U$, $\mathrm{Bd}(U)$ is a simple curve, $\mathrm{Bd}(U) \cap f^{-1}(t_1) = \{x_1, x_2\}$ and x belongs to an arc contained in $f^{-1}(t_1)$ with endpoints x_1 and x_2. Then, by the Theorem About The θ-Curve [9, Theorem 61.II.2], $U \setminus f^{-1}(t_1)$ has exactly two components, and consequently x belongs to the boundary of exactly two components of $\mathbb{R}^2 \setminus f^{-1}(t_1)$. Since f is co-Lipschitz, it is not possible that both of these components are contained in $f^{-1}(t_1, \infty)$, which contradicts (5.16) and ends the proof of (5.15) and of part (b).

For part (c), let $\{A_\nu\}_{\nu=1}^{m}$ denote the collection of components of $f^{-1}(t_2, \infty)$. As above, let $\mathcal{R}_{A_\nu}(t_1)$ denote the set of rays of $f^{-1}(t_1)$ contained in A_ν. By (5.14), for any $\nu \in \{1, \ldots, m\}$,

$$\#e(f^{-1}(t_1) \cap A_\nu) = \mathrm{card}(\mathcal{R}_{A_\nu}(t_1)). \tag{5.17}$$

Let $\gamma : \mathcal{R}(t_1) \longrightarrow \mathcal{R}(t_2)$ be the map defined in part (a). We will show that for all $\nu \in \{1, \dots, m\}$,

(5.18) $$X \in \mathcal{R}_{A_\nu}(t_1) \quad \Longrightarrow \quad \gamma(X) \subset \mathrm{Bd}(A_\nu).$$

Indeed, let $X \in \mathcal{R}_{A_\nu}(t_1)$ and $x \in X$ with $\|x\| \geq R_0 + 2|t_1 - t_2|$. By (5.12), there exists $y \in \gamma(X) \setminus B(0, R_0)$ so that $d(x, y) < 2|t_1 - t_2|$. Let I denote the interval with endpoints x and y. If $y \notin \mathrm{Bd}(A_\nu)$, then $I \cap \mathrm{Bd}(A_\nu) \neq \emptyset$, since $\mathrm{Bd}(A_\nu)$ separates between x and y. Thus there exists $y_1 \in I \cap \mathrm{Bd}(A_\nu)$ so that $y_1 \in f^{-1}(t_2)$, $\|y_1\| \geq R_0$ and $d(y, y_1) < 2|t_1 - t_2|$. Thus, by (5.10), $y_1 \in \gamma(X)$ and $\gamma(X) \cap \mathrm{Bd}(A_\nu) \neq \emptyset$. Hence, by (5.15), $\gamma(X) \subset \mathrm{Bd}(A_\nu)$ and (5.18) holds. If $y \in \mathrm{Bd}(A_\nu)$ the same argument proves (5.18).

Since γ is one-to-one, (5.18) immediately implies that

(5.19) $$\mathrm{card}(\mathcal{R}_{A_\nu}(t_1)) \leq \#e(\mathrm{Bd}(A_\nu)).$$

Since sets $\{A_\nu\}_{\nu=1}^m$ are disjoint, by (5.8) and by part (b) applied to t_2 and the set $A = \mathbb{R}^2$, we get

$$\begin{aligned}\#e(f^{-1}(t_1)) = \mathrm{card}(\mathcal{R}(t_1)) &= \sum_{\nu=1}^{m} \mathrm{card}(\mathcal{R}_{A_\nu}(t_1)) \\ &\leq \sum_{\nu=1}^{m} \#e(\mathrm{Bd}(A_\nu)) = \#e(f^{-1}(t_2)).\end{aligned}$$

Since, by part (a), $\#e(f^{-1}(t_1)) = \#e(f^{-1}(t_2))$, we conclude that, for all $\nu \in \{1, \dots, m\}$, $\#e(f^{-1}(t_1) \cap A_\nu) = \mathrm{card}(\mathcal{R}_{A_\nu}(t_1)) = \#e(\mathrm{Bd}(A_\nu))$, which ends the proof of part (c). □

For the proof of the main theorem we will need one more lemma.

Lemma 5.14. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a co-Lipschitz uniformly continuous map, $t \in \mathbb{R}$ and A be a component of $f^{-1}(t, \infty)$. Then:*

(a) *for every $s > t$,*
$$\#c(f^{-1}(s) \cap A) \leq \#c(f^{-1}(t) \cap \mathrm{Bd}(A));$$
(b) *there exists $\varepsilon > 0$ so that for every $s \in (t, t + \varepsilon)$,*
$$\#c(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \mathrm{Bd}(A));$$
(c) *let $t_1 > t$ and let $\{C_\nu\}_{\nu=1}^k$ be components of $f^{-1}(t_1, \infty)$ which are contained in A, then*
$$\sum_{\nu=1}^{k} \#c(f^{-1}(t_1) \cap \mathrm{Bd}(C_\nu)) = \#c(f^{-1}(t) \cap \mathrm{Bd}(A));$$
(d) *let $m = \#c(f^{-1}(t) \cap \mathrm{Bd}(A))$, then there exists a set $T_A \subset (t, \infty)$ with $\mathrm{card}(T_A) \leq m - 1$, so that for every $s \in (t, \infty) \setminus T_A$,*
$$\#c(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \mathrm{Bd}(A)).$$

Proof. Let $s > t$. By Lemma 5.13(c) and Corollary 5.7,
$$\#e(f^{-1}(s) \cap A) = \#e(\mathrm{Bd}(A)) = 2\#c(f^{-1}(t) \cap \mathrm{Bd}(A)).$$

Since, by Proposition 5.6, each component of $f^{-1}(s) \cap A$ has at least 2 ends, we get that
$$\#c(f^{-1}(s) \cap A) \leq \frac{1}{2}\#e(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \mathrm{Bd}(A)),$$
and part (a) is proven.

For part (b) we assume, without loss of generality, that the co-Lipschitz constant of f is equal to 1. Denote $l = \#c(f^{-1}(t) \cap \mathrm{Bd}(A))$, and let $\{L_j\}_{j=1}^l$ be the components of $f^{-1}(t)\cap\mathrm{Bd}(A)$. By Proposition 5.6, the intersection of any component of $f^{-1}(t)$ with $\mathrm{Bd}(A)$ is connected and therefore each L_j is contained in a different component of $f^{-1}(t)$. Hence, by Corollary 5.11, there exists $\delta > 0$ so that, for all $i, j \in \{1, \dots, l\}, i \neq j$, $d(L_i, L_j) \geq \delta$. Define for $j \in \{1, \dots, l\}$,

$$U_j \stackrel{\text{def}}{=} \bigcup_{x \in L_j} B(x, \frac{\delta}{2}) \cap A.$$

Then $\{U_j\}_{j=1}^l$ are connected, mutually disjoint, open subsets of A.

We note that, for any $y \in \mathbb{R}^2$,

$$d(y, f^{-1}(t)) \geq \frac{\delta}{2} \quad \Longrightarrow \quad |f(y) - t| \geq \frac{\delta}{2}. \tag{5.20}$$

Indeed, if $d(y, f^{-1}(t)) \geq \frac{\delta}{2}$, then $t \notin f(B(y, \frac{\delta}{2}))$. But, since f is co-Lipschitz with constant 1, $f(B(y, \frac{\delta}{2})) \supset B(f(y), \frac{\delta}{2})$. Thus $t \notin B(f(y), \frac{\delta}{2})$ and (5.20) holds. Hence, for all $s \in (t, t+\frac{\delta}{2})$,

$$f^{-1}(s) \cap A \subset \bigcup_{j=1}^{l} U_j. \tag{5.21}$$

Now fix $j_0 \in \{1, \dots, l\}$, and let $x \in L_{j_0}$ and $y \in \mathrm{Bd}(U_{j_0}) \setminus \mathrm{Bd}(A)$. Then $d(y, f^{-1}(t)) \geq \frac{\delta}{2}$.

Moreover, since L_{j_0} is locally connected and U_{j_0} is open, $\overline{U_{j_0}}$ is arcwise connected and there exists a continuous function $\sigma : [0, 1] \longrightarrow \overline{U_{j_0}}$ so that $\sigma(0) = x, \sigma(1) = y$ and $\sigma(\lambda) \in U_{j_0}$ for $\lambda \in (0, 1)$. Define $g : [0, 1] \longrightarrow \mathbb{R}$ as $g = f \circ \sigma$. Then

$$\begin{aligned} g(0) &= f(x) = t, \\ g(1) &= f(y) \geq t + \frac{\delta}{2}, \quad \text{by (5.20)}. \end{aligned}$$

By the Intermediate Value Theorem, for every $s \in (t, t + \frac{\delta}{2})$, there exists at least one $\lambda_s \in (0, 1)$ so that $s = g(\lambda_s) = f(\sigma(\lambda_s))$. Since $\sigma(\lambda_s) \in U_{j_0}$, we conclude that, for every $j_0 \in \{1, \dots, l\}$ and every $s \in (t, t + \frac{\delta}{2})$, $f^{-1}(s) \cap U_{j_0} \neq \emptyset$. Since sets $\{U_j\}_{j=1}^l$ are mutually disjoint and by (5.21), we get that for all $s \in (t, t + \frac{\delta}{2})$,

$$\#c(f^{-1}(s) \cap A) \geq l = \#c(f^{-1}(t) \cap \mathrm{Bd}(A)).$$

This, together with part (a), concludes the proof of part (b).

Part (c) follows by the following computation:

$$\begin{aligned} \sum_{\nu=1}^{k} \#c(f^{-1}(t_1) \cap \mathrm{Bd}(C_\nu)) &= \frac{1}{2} \sum_{\nu=1}^{k} \#e(\mathrm{Bd}(C_\nu)), && \text{by Corollary 5.7,} \\ &= \frac{1}{2} \#e(f^{-1}(t_1) \cap A), && \text{by Lemma 5.13(b),} \\ &= \frac{1}{2} \#e(\mathrm{Bd}(A)), && \text{by Lemma 5.13(c),} \\ &= \#c(f^{-1}(t) \cap \mathrm{Bd}(A)), && \text{by Corollary 5.7.} \end{aligned}$$

To prove part (d), we proceed inductively with respect to m.

If $m = 1$, then by part (a), for every $s \in (t, \infty)$,

$$\#c(f^{-1}(s) \cap A) \leq 1,$$

and since $f(A) = (t, \infty)$, we have $f^{-1}(s) \cap A \neq \emptyset$, and hence $\#c(f^{-1}(s) \cap A) \geq 1$. Therefore part (d) holds with $T_A = \emptyset$, as desired. For the induction, we assume that part (d) holds for all $m < m_0$, where $m_0 \geq 2$. Now suppose that $\#c(f^{-1}(t) \cap \mathrm{Bd}(A)) = m_0 \geq 2$. Define

$$t_1 \stackrel{\text{def}}{=} \sup\{\tau \in (t, \infty) : \forall s \in (t, \tau) \ \ \#c(f^{-1}(s) \cap A) = \#c(f^{-1}(t) \cap \mathrm{Bd}(A))\}.$$

If $t_1 = \infty$ there is nothing to prove, so suppose that $t_1 < \infty$. By part (b), $t_1 > t$ and $\#c(f^{-1}(t_1) \cap A) \neq \#c(f^{-1}(t) \cap \mathrm{Bd}(A))$. By part (a), this implies that

$$\#c(f^{-1}(t_1) \cap A) < \#c(f^{-1}(t) \cap \mathrm{Bd}(A)). \tag{5.22}$$

Let $\{C_\nu\}_{\nu=1}^k$ denote all components of $f^{-1}(t_1, \infty) \cap A$. Then, by (5.22), for each $\nu \leq k$,

$$\#c(f^{-1}(t_1) \cap \mathrm{Bd}(C_\nu)) \leq \#c(f^{-1}(t_1) \cap A) < \#c(f^{-1}(t) \cap \mathrm{Bd}(A)) = m_0. \tag{5.23}$$

Hence, by the inductive hypothesis, for each $\nu \leq k$ there exists a set $T_\nu = T_{C_\nu} \subset (t_1, \infty)$ with $\mathrm{card}(T_\nu) \leq \#c(f^{-1}(t_1) \cap \mathrm{Bd}(C_\nu)) - 1$, so that for every $s \in (t_1, \infty) \setminus T_\nu$,

$$\#(f^{-1}(s) \cap C_\nu) = \#c(f^{-1}(t_1) \cap \mathrm{Bd}(C_\nu)). \tag{5.24}$$

Set $T_A = \bigcup_{\nu=1}^k T_\nu \cup \{t_1\}$. Then for every $s \in (t_1, \infty) \setminus T_A$, we have:

$$\begin{aligned}
\#c(f^{-1}(s) \cap A) &= \sum_{\nu=1}^{k} \#c(f^{-1}(s) \cap C_\nu) = \sum_{\nu=1}^{k} \#c(f^{-1}(t_1) \cap \mathrm{Bd}(C_\nu)), && \text{by (5.24)}, \\
&= \#c(f^{-1}(t) \cap \mathrm{Bd}(A)), && \text{by part (c)}.
\end{aligned}$$

To finish the proof we only need to estimate the cardinality of the set T_A. We have

$$\begin{aligned}
\mathrm{card}(T_A) &\leq \sum_{\nu=1}^{k} \mathrm{card}(T_\nu) + 1 \\
&\leq \sum_{\nu=1}^{k} \left[\#c(f^{-1}(t_1) \cap \mathrm{Bd}(C_\nu)) - 1\right] + 1 \\
&= \#c(f^{-1}(t) \cap \mathrm{Bd}(A)) + (1 - k), && \text{by part (c)}. \\
&\leq \#c(f^{-1}(t) \cap \mathrm{Bd}(A)) - 1, && \text{since, by part (c) and (5.23), } k \geq 2,
\end{aligned}$$

which ends the proof of part (d). □

We are now ready for the proof of our main theorem.

Proof of Theorem 5.1. To prove part (1a) we note that by Corollary 5.9, number of ends of any level set $f^{-1}(t)$, for $t \in \mathbb{R}$, is finite and bounded by a constant M depending only on the co-Lipschitz constant of f and its modulus of uniform continuity. By Lemma 5.13(a), $\#e(f^{-1}(t))$ does not depend on the value of $t \in \mathbb{R}$. To see that $\#e(f^{-1}(t))$ is even, let

$\{A_\lambda(t)\}_{\lambda=1}^l$ be the components of $\mathbb{R}^2 \setminus f^{-1}(t)$. Then we have

$$\begin{aligned}
\#e(f^{-1}(t)) &= \sum_{\lambda=1}^{l} \#e(\mathrm{Bd}(A_\lambda(t))), && \text{by Lemma 5.13(b),}\\
&= 2\sum_{\lambda=1}^{l} \#c(f^{-1}(t) \cap \mathrm{Bd}(A_\lambda(t))), && \text{by Corollary 5.7.}
\end{aligned}$$

Thus $\#e(f^{-1}(t))$ is even.

Part (1b) follows from Corollary 5.12, and hence part (1) is proven.

For the proof of part (2), let t_0 be any real number, say $t_0 = 0$, and let $\{A_\nu\}_{\nu=1}^l$ be all the components of $f^{-1}(0,\infty)$. By Lemma 5.14(d), for every $\nu \le l$, there exists a set $T_{A_\nu} \subset (0,\infty)$ with $\mathrm{card}(T_{A_\nu}) \le \#c(f^{-1}(0) \cap \mathrm{Bd}(A_\nu)) - 1$, so that for all $s \in (0,\infty) \setminus T_{A_\nu}$,

$$\#c(f^{-1}(s) \cap A_\nu) = \#c(f^{-1}(0) \cap \mathrm{Bd}(A_\nu)). \tag{5.25}$$

Define

$$T_+^0 = \bigcup_{\nu=1}^{l} T_{A_\nu}.$$

Note that

$$\begin{aligned}
\sum_{\nu=1}^{l} \#c(f^{-1}(0) \cap \mathrm{Bd}(A_\nu)) &= \frac{1}{2}\sum_{\nu=1}^{l} \#e(\mathrm{Bd}(A_\nu)), && \text{by Corollary 5.7,}\\
&= \frac{1}{2}\#e(f^{-1}(0)), && \text{by Lemma 5.13(b).}
\end{aligned} \tag{5.26}$$

Therefore for every $s \in (0,\infty) \setminus T_+^0$, we have:

$$\begin{aligned}
\#c(f^{-1}(s)) = \sum_{\nu=1}^{l} \#c(f^{-1}(s) \cap A_\nu) &= \sum_{\nu=1}^{l} \#c(f^{-1}(0) \cap \mathrm{Bd}(A_\nu)), && \text{by (5.25),}\\
&= \frac{1}{2}\#e(f^{-1}(0)) = n, && \text{by (5.26) and part (1).}
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{card}(T_+^0) \le \sum_{\nu=1}^{l} \mathrm{card}(T_{A_\nu}) &\le \sum_{\nu=1}^{l} \left[\#c(f^{-1}(0) \cap \mathrm{Bd}(A_\nu)) - 1\right]\\
&= \frac{1}{2}\#e(f^{-1}(0)) - l \le n - 1, && \text{by (5.26) and part (1).}
\end{aligned}$$

Next, we note that in an identical way (e.g. by replacing function f by $-f$) one can define a set $T_-^0 \subset (-\infty, 0)$ with $\mathrm{card}(T_-^0) \le n-1$, so that for every $s \in (-\infty,0) \setminus T_-^0$, $\#c(f^{-1}(s)) = n$. Define

$$T_f^0 = T_+^0 \cup T_-^0 \cup \{0\}.$$

Clearly, $\mathrm{card}(T_f^0) \le 2n-1$. Now, let $t_{00} \in \mathbb{R}$ be such that $t_{00} < t$ for every $t \in T_f^0$. Then we clearly have $\#c(f^{-1}(s)) = n$, for every $s \le t_{00}$. Same way as was done above for $t_0 = 0$, we construct a set $T_+^{00} \subset (t_{00},\infty)$ so that for every $s \in (t_{00},\infty) \setminus T_+^{00}$, we have $\#c(f^{-1}(s)) = n$ and $\mathrm{card}(T_+^{00}) \le n-1$. Then part (2a) holds for $T_f \overset{\mathrm{def}}{=} T_+^{00}$.

Part (2b) follows immediately by Corollary 5.8.

For part (2c), note that by Proposition 5.6, each component of $f^{-1}(t)$ has at least 2 ends. Since by parts (1) and (2a) for all $t \in \mathbb{R} \setminus T_f$, $f^{-1}(t)$ has n components and $2n$ ends, we conclude that each component of $f^{-1}(t)$ has exactly 2 ends. Hence, by Proposition 5.6, each component K of $f^{-1}(t)$ separates the plane into exactly 2 components. Further, by Theorem 4.11, K has a representation of the form $K = K_0 \cup K_1 \cup K_2$, where $K_1 \cap K_2 = \emptyset$, and for $i = 1, 2$, $K_i \cap K_0$ consists of exactly one point which is an endpoint of both K_i and K_0, K_1 and K_2 are both homeomorphic with $[0, \infty)$, and K_0 is a compact connected tree with exactly 2 endpoints. Thus K_0 is homeomorphic with $[0, 1]$ (either by the construction of K_0 described in the proof of Theorem 4.11, or by the classical characterization of the interval as a continuum with exactly 2 non-cut points cf. e.g. [20, Theorem III.(6.2)]). Therefore K is homeomorphic with $(-\infty, 0] \cup [0, 1] \cup [1, \infty) = (-\infty, \infty)$, which ends the proof of part (2c).

For part (3a) we note that for all $t_i \in T_f$, $\#c(f^{-1}(t_i)) \neq n$. Since $\#e(f^{-1}(t_i)) = 2n$ by part (1), and each component has at least 2 ends, by Proposition 5.6, we conclude that $\#c(f^{-1}(t_i)) < n$, i.e. part (3a) holds. Parts (3b) and (3c) follow immediately from Corollary 5.8 and Theorem 4.11, respectively. □

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Department of Mathematics and Statistics, Miami University, Oxford, OH 45056
E-mail address: `randrib@muohio.edu`